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ABSTRACT

This is a SMSG inservice textbook for primary school teachers. One of the goals of the book is to promote the teaching of mathematics in accord with a conceptual development of mathematical ideas. The authors indicate that rote learning is frequently considered the only way to learn mathematics, especially for culturally deprived children. A feature of this text is an attempt to attend to learning problems that may be associated with the culturally disadvantaged. Other features of the text include sections in each chapter dealing with applications to teaching and frequently asked questions. Chapter topics include: (1) description of culturally disadvantaged children; (2) sets; (3) comparing sets; (4) whole numbers; (5) set operations; (6) introduction to geometry; (7) numeration-naming numbers; (8) addition; (9) multiplication; (10) subtraction; (11) division; (12) elements of geometry; (13) addition and subtraction techniques; (14) introducing rational numbers; (15) premeasurement concepts; (16) multiplication and division techniques; (17) measurement; and (18) structure. (MP)

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STUDIES IN MATHEMATICS

VOLUME XIII

Inservice Course in Mathematics for Primary School Teachers

(revised edition)

SE 025 373

STUDIES IN MATHEMATICS

Volume XIII

INSERVICE COURSE IN MATHEMATICS FOR PRIMARY SCHOOL TEACHERS

(revised edition)

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Preface

A sound knowledge of mathematics is becoming a prerequisite for fruitful work in an ever-increasing number of endeavors. This knowledge must include why mathematical processes work as well as how they work. It is not enough for today's children to learn mathematics by rote.

Children now in elementary shcool can be expected to face problems which we cannot foresee. These problems will be solved not by knowledge of mathematical facts alone, but by knowledge of mathematical methods for attacking problems. New and as yet unknown problems may involve, and in fact will require, new and as yet unknown mathematics for their solutions. Naturally, we cannot teach this unknown mathematics, but we can and must teach methods of mathematical thinking as well as the basic content of mathematics.

In general, schools today are becoming increasingly aware of the need to orient the teaching of mathematics in accord with a conceptual development of mathematical ideas. Yet, too frequently an assumption is made that for a population of children that is considered to be "culturally deprived", rote learning is still the only answer to learning mathematics. This course of action would further deprive these children. A feature of this text is an attempt to attend to problems that may be associated with the culturally disadvantaged.

The introductory chapter begins with a description of the culturally disadvantaged based on psychological findings. It continues with the physical, social, and psychological environment in which these children function in their pre-school years. The next concern is with the characteristics manifest in the culturally disadvantaged children as they enter school. Finally, implications for teaching these children are discussed in this chapter.

Chapters 1 through 17 present mathematical content relevant to teaching in the primary grades. All topics which are included in the texts for the School Mathematics Study Group Books K and 1 are treated, but from a more sophisticated point of view. Other topics have been included when the development has warranted it. As Book K and 1 do not develop topics in the same sequence, adaptation of this course to a particular grade may necessitate some minor change in the order of studying these materials. The two tables below identify each chapter in Book K and in Book 1 with the chapter in this book that pertains to the same topics. Thus, the sequence of the chapters in the inservice text listed in the right hand column might serve as a guide for the order in the studying.

Chapter in Book K	Inservice Chapter
1. Sets	1
2. Recognizing Geometric Figures	5
3. Comparison of Sets	2
4. Subset of a Set	1
5. Joining and Removing	4
6. Comparison of Sizes and Shapes	14, 16
7. Ordering of Sets	2
8. Additional Activities (geometric shapes)	11
9. Using Numbers with Sets	3, 6

Chapter in Book 1	Inservice Chapter
1. Sets and Numbers	1, 2
2. Numerals and the Number Line	3, 5
3. Sets of Ten	6
4. Introduction to Addition and Subtraction	4
5. Recognizing Geometric Figures	5, 11
6. Place Value and Numeration	6
7. Addition and Subtraction	7, 9
8. Arrays and Multiplication	8
9. Partitions and Rational Numbers	13
10. Linear Measurement	14, 16

Both Books K and 1 start with the notion of set, a primitive notion on which other mathematical concepts will be built. After this development, the orders in presentation of topics differ considerably. The reason for the difference is largely in consideration of concept development of children at the different levels. For example, in the kindergarten program, it makes sense to start with activities associated with geometric concepts since these activities involve working with familiar objects such as boxes and tin cans. These activities do not require as high a level of abstraction as does the number concept. Moreover, they are activities that encourage the learning of sorting and classifying of objects and ideas, an implicit, if not explicit requirement for learning numbers in particular, and mathematics in general. On the other hand, the first grade course starts with the concept of number right after the development of sets because the number concept is at hand by then. Book 1 then builds upon the preliminary groundwork laid in the kindergarten course and extends the concept of numbers to those greater than 9.

Another example illustrating differences in sequence may be seen in the order of presenting the arithmetic operations. In Book 1, subtraction immediately follows addition, whereas in the inservice text, multiplication follows addition. The order as presented in Book 1 is the one by which children usually learn these operations; the procedure adopted for the inservice text discusses first the primary operations of addition and multiplication and then brings into focus the secondary operations for subtraction and division as the inverse operations of addition and multiplication respectively.

The remainder of the book consists of three appendices containing background information. Appendix A is a description of the SMSG mathematics program for grades K-3. Here, is displayed a chart showing the scope and organization of mathematical contents in these grades. The inclusion of such a chart is intended to provide perspective to the teaching of mathematics in the elementary school by showing when a certain topic occurs, how its occurence is related to other topics in the sequence, and when it recurs again in the spiral of the curriculum.

Appendix B attends to language and mathematical learning. The careful building of understanding and correct use of mathematical language through aural-oral experiences is considered. Particular examples and suggestions useful to teachers of young children are included.

Appendix C contains information gathered from observations and testing of children who used the School Mathematics Study Group texts, MATHEMATICS FOR THE ELEMENTARY SCHOOL, Books K and 1 during the school year 1964-65. These results are taken from a comparative study of children described in the introductory chapter, DESCRIPTION OF CULTURALLY DISADVANTAGED CHILDREN.

This book has been written with an inservice course in mind; however, it is hoped that the text is sufficiently lucid to be understood easily by the reader. It is assumed that greater comprehension and interest will derive from discussion between an instructor and inservice teachers. For this purpose, the problems that have been inserted at appropriate intervals in each chapter (as opposed to a set of exercises at the end of each chapter) may be an integral part of disucssions for clarifying fine points and for deepening understanding.

When it was felt that some comment on pedagogy or other relevant remarks might contribute toward better understanding and teaching of the concepts, these comments are included under the section, APPLICATIONS TO TEACHING, at the end of the chapter. From inservice meetings and other contacts with teachers in the primary grades, a few questions pertaining to various topics have been observed to recur. We hope that by selecting some of these frequently asked questions when they are relevant to the chapter and expanding on the underlying concepts, we can resolve some of the difficulties that may have arisen. For want of a better handle, we shall label such sections, QUESTION. It is important to note that a large part of what is presented here is background material for you, as a teacher, and is not intended to be transmitted to your students per se. We hope that as you read the text and do the exercises, you will increase your understanding of some basic notions underlying the mathematics that you are teaching, and that, in general, this text will prove useful.

In the preparation of this book, we have drawn on materials produced by the School Mathematics Study Group, and in particular, STUDIES IN MATHEMATICS, Volume IX, A BRIEF COURSE IN MATHEMATICS FOR ELEMENTARY SCHOOL TEACHERS. For the use of these materials, we offer acknowledgement.

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Chapter 0

DESCRIPTION OF CULTURALLY DISADVANTAGED CHILDREN

I. Introduction

From a variety of sources, data have been accumulating which explain the disadvantaged position of the culturally deprived child as he starts school. If we review some of the conditions within the family and neighborhood which these children experience during their pre-school years, then the characteristics of these disadvantaged children become more meaningful.

The two major criteria used in defining people classified as culturally disadvantaged are (1) low economic status, and (2) lack of participation in middle-class culture. The actual family income may vary from one study to another. A maximum family income of $2,000 per year defines the disadvantaged in some studies; an income below $4,000 may define this group for others working with them.

The criterion of lack of participation in middle-class culture is more difficult to specify, but relates most closely to the values placed upon education. The lack of books, of parental examples of reading and success in education, and the lack of stimulation to achieve are all parts of this non-participation in middle-class culture.

The culturally disadvantaged group consists mainly of urban slum-dwelling people, particularly because the United States population is becoming increasingly more urban. This fact does not, however, preclude inclusion of such marginal subsistence groups as segregated rural Negroes, dwellers in the depressed areas of Appalachia, and many American Indian groups from the ranks of those described as culturally disadvantaged.

II. Contributing Factors

If we look to the home and environmental circumstances that influence these children in ways which are apparent at school entry, the physical living conditions as well as the quality of the parent-child interactions are most striking. There are, of course, exceptions to these observations so that rash generalizations should not be made. We mention some of the more salient observations in order to alert the reader to these factors.

A. Physical Living Conditions

The living conditions provide a particular kind of setting within which the parent-child interactions take place. The crowding of dwellings in disadvantaged areas of large cities allows for little privacy, solitude, or quiet. Not only are there likely to be many people occupying a small apartment, but the dwelling units are close upon each other. In other words, the density of people for the physical space is very high. What this means for a young child is that he has almost no place to play without being either "underfoot" or out on the street. He is constantly subjected to noise from the family, television, neighboring households, and street activity.

The child in the disadvantaged home is not likely to have books or magazines available to look at nor to have read to him. He is not likely to have a variety of toys with which he can amuse himself nor toys which encourage sharing. The possibility of developing gross and fine muscle coordination and independent imaginative play through drawing, cutting, and building blocks, for example, are lacking. He is less likely to have been taken on trips outside the immediate neighborhood--to the zoo, parks, a farm, museums, or even to the library. Thus, the experiences of these children prior to school entry have been different from those of children of middle-class families and much more highly restricted in variety.

B. Hostility of the Environment

In the above section the typical household situation was described. The character of the neighborhood, as the broader social setting, also influences these children in ways which are apparent at school entry.

The environment of the disadvantaged is described as hostile because of the higher rates of delinquency, disease, and death in their neighborhoods.[1] Whether or not these conditions can be called hostile, the following conditions are, at the least, not conducive to moving outside the home or relating to the community. First, fewer public recreational facilities are located in these areas than in areas of higher income residences.

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[1] For studies supporting these and the immediately following statements, see Sexton, Patricia C. Education and Income. New York: The Viking Press, 1961.

Second, school buildings in these areas tend to have less adequate equipment and facilities for such activities as science and art, to mention two of many which could be cited. Third, contacts outside the immediate neighborhood have frequently been with authority figures--policemen, welfare workers, or school officials. The school is likely to be associated with this authority and resented rather than considered an important resource for help and development of potential.

C. Parent-Child Relationships

Since the basic family unit among many of the disadvantaged groups does not consist of mother, father and their children, the effects of other compositions of the family unit must be considered. In many of these homes the father is not present. The household often consists of mother, children, and possibly other female adults such as an aunt or maternal grandmother. There may be considerable instability both in the living arrangements and in the adults important to the child. For example, a maternal aunt and her children may move into the household if there is a crisis in their lives; or a child may leave his mother and move to a relative's home if his mother takes a job.

What emerges is a form of "extended family" which provides a certain safety and security against what may be perceived as the hostile world. What lack appears most significant, especially for the child's school performance, is that of a directed interaction between the adult and the child. The mother in the culturally disadvantaged home is not likely to spend time in conversing with one of her children alone, nor in sitting down to teach him a specific skill such as tying his shoelaces. Supervision of the child is handled by any of the adults available, by older siblings, or none.

To clarify the term "adult-child interaction", let us use two typical situations in a two-year-old's exploration of his environment. He may reach for something hot, or poke a finger into the eye of his baby sister. Mother, who is likely to be preoccupied with the sheer physical demands of life, does not explain why the child's behavior will be harmful to him or the baby; rather she will yell at him, "Stop that!", or "Bad boy!", or simply slap him.

The implications of mother's response for the two-year-old are several. First, there is no verbal specification of what, exactly, the undesired act was. He may interpret mother's slap as meaning that *reaching* or *touching* is wrong and, therefore, punishable. Thus, there is no opportunity for learning a discrimination between the *act* of reaching or touching and the *consequences* of reaching for *certain* objects (in this instance, hot) or touching (the baby's eye). What this kind of punishment is likely to achieve, if used as the usual means of discipline, is a stifling of reaching and touching. This will eventually diminish the child's curiosity by reducing his explorations of his immediate environment.

A second implication of mother's use of "Stop that!" or a slap is that it does not provide a model for complex verbal behavior. The child needs to listen to language forms in order to pattern his own language, both in terms of range of vocabulary and complexity of expression. Also, he needs the experience of verbally expressing his questions, reasons, and feelings in order to learn to communicate verbally.

A third implication of mother's response in the earlier cited examples is for the child's self-concept. If mother had said in response to his reaching or poking, "What a big boy you are to be able to reach so high (onto the stove or into the crib)! But you must be careful about hot things or baby's eyes," then there is some increment in a positive concept of himself. He is growing and is capable of new accomplishments. By slapping or telling him that he is a bad boy, however, his image of himself is deflated. What is likely to evolve in this setting is an image of the good child as one who stays out of the way and who is quiet. This is not the child who will achieve in school through high motivation and striving.

A further point should be made about the relationships between mother and child as contributing to the child's behavior as we see him at school entrance. If the mother has not had many years of formal education herself, she will be less aware of what experiences she could provide which would eventually help her child in school.

D. Planning and Scheduling of Time

A characteristic of many disadvantaged families, partly related to their living conditions and partly related to their sub-culture, is the lack of a family schedule or routine. Meals are not eaten at regular times, nor is there a set bedtime for the children. It is seldom that the family sits down to a meal together.

There are two effects of this lack of time-planning and routine which are likely to cause difficulties in the young child's adjustment to school. The first has to do with adapting to a routine and working independently within time limits. The second has to do with verbal development.

Let us deal, first, with the use of time. The child who has not experienced some scheduling of activities at home will not be able, without considerable help, to adapt to a school routine--a time to start a given activity with the class and a time to finish up that activity when directed. This means less self-direction and less ability to work independently. In addition, being on time has no meaning unless expectations have been established that certain events occur at particular times and some consequences may follow from not being on time.

Without the experience of planning time and using periods of time within the day for particular activities, the child is less likely to be able to develop longer-range, more abstract goals which involve both planning longer blocks of time and sequencing time. It may well be that the demands of career plans involving particular steps both in the immediate and more remote future are not possible without these early experiences. Successful performance in specially selected courses in high school, along with summer jobs to earn money, in order to enter college involves such sequenced, long-term planning. When a mother explains to her pre-school child that he may play with his friend at a certain time, after his nap, or when she says that he may watch television until supper at 5:30, she may be laying the foundation for later longer-range, goal-oriented planning.

The fact that the family does not sit down together for a meal or for discussing the day's happenings permits fewer opportunities for verbal interaction. The child does not have the experience of hearing, attending to, or participating in complex verbal expression. The child who does not participate with adults in such verbal interchange has little opportunity to be heard, to be corrected, and therefore to have his language modified and expanded. At school entrance, the child comes into a situation where there are expectations that he express his ideas in verbal interchange with an adult. The situation is strange and unfamiliar.

He must learn to adapt to this new kind of interaction as well as to learn the language necessary in order to participate.

In discussing the mother-child interactions earlier in this chapter, it was pointed out that the mother does not use complex verbal explanations in directing her child's behavior. More physical, rather than verbal, means of discipline, plus the lack of conversation among family members, combined with the lack of direct teaching, contribute to the development of a child experientially limited in both the content (vocabulary, variety, and complexity of speech forms) and the structure of such verbal communication.

E. Lack of Preschool Experience

A source of enrichment for some young children, though usually not for the deprived child, has been a year or more of nursery school prior to school entry. Since nursery schools have traditionally been privately funded, therefore requiring tuition, they have not, in the past, been available to the disadvantaged groups. With the increasing governmental concern for the economically deprived segments of our population, such programs as Project Headstart will undoubtedly have influence on the experiential development of these children.

III. Characteristics of Culturally Disadvantaged Children

In the two preceding sections we have described certain characteristics of culturally disadvantaged families which influence their children's behavior by the time they start school. In this section, we shall describe feelings and behavior of these children in the beginning school years resulting from the family and broader environmental influences.

A. Self Concept

Given the conditions of a hostile environment, of their families being the "have-nots" economically and socially, and of the lack of experiences directly relevant to classroom learning, these children are not likely to have positive feelings about themselves nor of society. Most crucial in the context of this book are their feelings about their competence for succeeding in school. For the Negro child especially, the effects of prejudice, segregation, and inferior status are likely to lead to negative feelings of his own worth.[2] These feelings, in turn, lead to little motivation or *striving* for success since these children are learning that their chances for success are relatively limited.

Since these children do not have the basic skills or know-how for immediate adaptation and successful performance in the classroom situation, they are likely to meet with frustration and confusion, if not failure, very quickly. Indeed, they may not be aware of instances in which they may have rejected avenues leading to success. The effects of such experiences will further detract from their feelings of competence.

Somewhat less tangible, but worthy of mention, is the observation that these children do not have as differentiated a self-concept as do more privileged children. By differentiation is meant the perception of one's self as a unique individual with certain characteristics, preferences, and wishes which form an identity distinguishing one from others. The reasons suggested for this lack of clearer differentiation are negative. That is, there is not an intense relationship between a parent and an individual child in these families, nor is the treatment of a given child individualized. As a result, these children display less self-concern, less competitiveness, and less motivation for self improvement. These are facets of intrinsic motivation which many teachers rely on to keep children in a given task.

There is a special problem in the development of a self-concept in boys from culturally disadvantaged homes. This arises from the lack of a stable father. As pointed out earlier, in many of these homes the father is absent. There may be adult males in and out of the home, but the presence of these potential models is likely to be temporary. This situation does not allow for a stable relationship with an adult male whom the boy may use as a model for imitation and identification. The adult males with whom the child does have contact are generally not those presenting a picture of responsibility or successful achievement as measured by the standards implicit within our schools.

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[2] For a detailed discussion of this topic, see Ausubel, D. and Ausubel, Pearl, "Ego development among segregated Negro children," in Passow, A. Harry, *Education in Depressed Areas*. New York: Teachers College, Columbia University, 1963.

B. Language

The language with which the culturally deprived child comes to school is likely to be different from the more advantaged child's in two major ways.[3] The first is in the quantity of verbal expression, and the second is in the quality of verbal expression.

Concerning quantity, children from disadvantaged homes tend to speak in short phrases. A monosyllabic response to a teacher's request or question is typical. It is not uncommon to see kindergarten children of this group sitting side by side at a table in a classroom and not having any verbal interchange at all. In certain pre-school enrichment programs it has been found necessary, in many instances, to encourage a child to express himself verbally by talking to an imaginary person over a toy telephone before he is able to speak directly with another child or with a teacher.

The difference between disadvantaged and more advantaged children is less likely to be seen in verbal labelling. That is, children from the disadvantaged groups can give names to commonly seen objects, such as a dog, pencil, box, key. If the object, or referent, is rarely experienced in low income environments, (e.g., "nest" which is experienced in rural environments or "giraffe" which requires books or a trip to the zoo), these words are not likely to be known by the children.

The major qualitative differences lie in the elaboration of languages--in complexity of grammatical expression and in the more abstract language which goes hand in hand with conceptual development. The experiences of these children have not prepared them well for simple classificatory behavior, such as comparing toys or other classroom materials on such dimensions as size, shape, color, number. They do not have the vocabulary for expressing such classifications or comparisons, nor have they had experiences which have made them attend to the attributes of objects. Children from the middle class, both more experienced verbally and more aware of abstracting from attributes, are better able to state a concept explicitly when given pictures all of which fit that concept. On the other hand, lower-class children use more concrete attributes and not necessarily the essential ones. To illustrate this idea, John[4] studied a group of Negro lower and middle-class first graders. She presented them with pictures of four men at work: a policeman, a doctor, a farmer, a sailor. The middle-class children more frequently said they were alike because all were "men" or "people", which are category labels. The lower-class children focused on non-essential attributes with such statements as "look the same", "like each other".

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[3] This discussion will not deal with the child from a non-English speaking home since this presents another kind of language problem beyond the scope of this section.

C. Sensory development

Lacking the experiences of attending to attributes of objects as discussed in the previous section and lacking the experiences of looking at books, these children are not likely to be as ready for the discrimination and attention demanded by printed materials. Teachers working with these children have found that they are easily confused about the task to be done by many pictures or numerals on a page; they quickly lose their places; one page in a workbook is readily confused with another which has similar elements.

Whether their eye-hand coordination is less well developed is not as crucial an issue here as the fact that they have not had the opportunities to use pencils, crayons, and scissors. Their experiences in seeing printed lines and pictures within books and finding meaning from them are limited.

In the area of auditory discrimination, the disadvantaged child does not attend as well to teacher directions nor to her instruction, probably for two reasons. First, he is easily distracted by extraneous sounds or activities. Secondly, and partly resulting from the first, he is less able to discriminate what is the sound to be listened to from the noise which impinges.

D. Motivation to achieve school goals

Considering the picture presented of the pre-school environment of a child from a disadvantaged group and knowing the expectations of the school for task-oriented activity, the discontinuity between the two settings is striking. The child has much to "unlearn", as well as to learn. He must learn the expectations of the school, and especially the demands of his teacher, without considering content learning at all!

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[4] John, Vera P. "The intellectual development of slum children: some preliminary findings." Amer. J. Orthopsychiat., 1963, 33, 813-822.

Much has been written on the antagonism and defensiveness of lower-class families toward the schools and of the discrimination (often unconscious and unintentional) on the part of the school in dealing with lower-class parents and children. The work of understanding such attitudes on both sides is not to be minimized. Quite apart from such factors, however, the difficulty in transition for the child entering school can readily be seen. He often lacks the long-range goals which can be achieved through school success, as well as the intrinsic motivation to learn for "self-improvement".

IV. Implications for Teaching

In the previous section of this chapter, a picture was sketched of the child from a disadvantaged group at the time he entered school. In this section, an attempt will be made to apply the understandings derived from the description of contributing factors and resulting characteristics to the teaching of disadvantaged children. In order to discuss teaching, references will be made to the performance of these children on school tasks and in test situations.

A. Implications for Teacher Attitude

It is very important for a teacher in the primary grades to be aware that the performance of these disadvantaged children in their early school years is not necessarily a good indication of their potential. Their earlier experiences have not well prepared them for the demands made by the school; therefore, they are not as ready for school. It should be kept in mind that their rate of learning can be very rapid on tasks which do not depend on prior learning that they have not had.

In this same vein, test results should be interpreted within the context of your knowledge of these children. Specifically, there are a number of facets of standardized tests and of the testing situation which contribute to their poorer performance. First, we can go back to the visual and auditory discrimination limitations discussed in Section III of this chapter. If the test directions are presented verbally, the child may not clearly understand what he is expected to do--assuming that he is able to attend sufficiently long to hear what is said. Then, assuming that the test is group administered, he is expected to use a pencil to mark certain symbols on a paper or booklet which involves both some dexterity in using the pencil and making rather fine discriminations among the symbols on the page. In addition to this, the relatively shorter attention span of these children compounds to the problem.

A test which is timed adds another factor contributing to the poorer performance of disadvantage children. These children are not accustomed to working within a time schedule as many middle-class children are.

Other factors that contribute to poorer test performances of this group are particularly relevant to intelligence tests. These are lack of practice or test "know-how", lack of motivation to do well, inadequate rapport with the examiner, and the cultural loading of the tests themselves which discriminates against these children.

This discussion of the factors influencing test performance is placed under the heading, "Implications for Teacher Attitude", for the purpose of increasing your awareness that a set of test scores does not permit accurate judgments of what the capacities, learning rates, or potential performance of disadvantaged children may be in the future. Such test scores give information on how a given child is performing at a particular point in time. How the same child might perform given opportunities to compensate for some of these limiting conditions is a major challenge to the schools at present, in attitude, as well as behavior.

The final admonishment concerning attitudes toward the culturally disadvantaged child is to keep in mind his earlier experiences which may make for difficulties in his relationship to you, as his teacher, in his adaptation to the school routine, and in his unfamiliarity with the work expected.

B. Implications for Teacher Behavior

Given some knowledge of the background and resultant characteristics of disadvantaged children, what can a teacher do to aid in their development and school progress? Four suggestions are given here which, it is hoped, will provide guidelines for your work and relationship to them.

The first suggestion, and perhaps the most important, is to maintain a warm and supportive relationship with these children. Although this may sound like an oft-stated platitude, it is particulary important for the groups that we are dealing with, and it can be acted upon in a variety of meaningful ways. In as many ways as possible, provide experiences which will enable the children to be successful. Conversely, avoid situations that may produce frustration and failure. These children need the reassurance that can be afforded by your attention and by regular and frequent praise. Their need for experiences in successful completion of tasks means that you must be careful and discriminating in what you ask each individual to do. For example, suppose you ask John to pair the members of two sets of objects at the flannel board, and he is unable to manage this task. You then ask Andy to go up to the flannel board to help John. Do not allow Andy to take over and complete the task. Make certain that what Andy does is helping, by doing one pairing only, and that it is John who actually completes the task successfully.

The variability in performance level of children in these classes, which will be discussed in Appendix C, makes it imperative that you deal with disadvantaged children individually and that you assure each child the experiences of completing his work, with expectations of success, at whatever his current level of performance.

The next two suggestions concern your own language and your encouragement of the children's verbal expression. Given their limitations in auditory discrimination and their inexperience with more complex language structure, they may not be able to understand your language easily. This will be particularly true if you use long, complex explanations or directions. They need short, simply stated directions until such time that they are able to understand more complex verbal expressions, feel more certain in their relationship to you, and you are sure they are able to perform what is being asked of them.

Complementary to the suggestion concerning your language is that of encouraging the children's verbal expression wherever possible. They need the experience of expressing, through words, their ideas and wishes to you as well as to the other children. There are many ways in which this can be accomplished; only a few will be suggested here. By asking children to describe objects (the objects being used for set construction, the toys they are playing with, the pictures they have drawn), you are both encouraging verbal expression and making them aware of attributes or properties (color, texture, size).

Another device which has been found very helpful is the use of word problems. If the teacher starts by telling stories which have problems in them which she expects the children to answer verbally, she can soon get the children to make up such problems for the class to answer.

The last suggestion, keeping the children involved, is certainly applicable to all children. It is especially important for the disadvantaged children. Your knowledge of these children will be of great assistance in this, as will the age of the children. With young children, you can use many sensory-motor experiences for teaching; there can be much more activity involving concrete materials from which abstractions can be made.

In summary, start your teaching at the level at which the children are able to function, use their assets, and maintain a high level of aspiration for yourself and for your pupils.

CHAPTER 1

SETS

INTRODUCTION

There are a few ideas that occur over and over again in mathematics; one of these is the concept of set. This concept occurs, for example, in dealing with sets of points, sets of numbers, sets of objects. The most general of these,of course, are sets of objects. From these sets we ultimately extract the concept of number. Thus, sets help form a primitive basis for the number concept and serve as pre-number ideas.

WHAT IS A SET

In speaking of collections of objects, special words may be used with reference to special collections such as:

herd of cattle (set of cattle),
flock of geese (set of geese),
pride of lions (set of lions),
navy (set of ships),
span of horses (set of horses).

Each of these may be equally described as a set; a set is just a collection of things. Some examples of sets of things are:

the furniture in a room,
the monkeys in the zoo,
the doors in a room,
the children in the class,
the books in the library.

Each object in a set is called a member or an element of the set. If the objects on your desk are a pencil, a book, a calendar, and a blotter, then each of these is a member of the set of things on your desk; each child in your class is a member of the set of children in your class.

A set may consist of a variety of objects. A prime characteristic of a set is that there is a method or rule whereby set membership or nonmembership can be determined. Consider the following examples.

1.) Suppose we consider the set of wheel toys. We ask the question, "Is a doll a member of the set?" Since a doll is not a wheel toy, it is not a member of the set. A wagon, on the other hand, is a member of the set, since it is a wheel toy.

2.) Suppose we consider the set of objects on the teacher's desk. The criterion for determining whether or not a particular object is a member of the set is, "Is this object on the teacher's desk?"

In both examples, there is a property that is shared by members of the set that is not shared by objects that are not members of the set. The common property of being a wheel toy, thus, is the rule that determines membership in the set in the first example. The common property of being on the teacher's desk is the rule that determines membership in the second set.

PROBLEMS*

1. What are the members of the set of
   a. the Great Lakes of the United States?
   b. the days of the week?
   c. the objects in Elsie's purse?
2. Determine which of the objects listed below are members of the set of animals.
   a. carrot
   b. lion
   c. tiger
   d. tree
   e. cat

DESCRIBING SETS

There are various ways in which a set may be specified. In the case of the set consisting of California, Oregon, and Washington, we may specify the set by listing all the members. A class roster is thus a means of specifying a particular set; a reading list is a means of specifying another set. If the reading list consists of the book titles, The Story of Ping, A Day in Maine, and Make Way for Ducklings, we can enclose these titles within braces { } to denote the set so specified. Thus,

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*Solutions for problems in this chapter are on page 27 .

{The Story of Ping, A Day in Maine, Make Way for Ducklings}

is a notation for "the set whose members are The Story of Ping, A Day in Maine, and Make Way for Ducklings." The braces are an abbreviation for the words "the set whose members are." Note that the items in the listing are separated by commas.

There are occasions when it is inconvenient or impractical to specify the set by listing all its members. For example, the set of all states of the United States requires a listing of 50 states; the set of all inhabitants in the United States may require a listing of more than 200 million names. If there is an explicit common property that may be used to characterize the members of the set, then such a description may be adequate. Thus,

{the states of the United States}

specifies the set being considered. For convenience, we may use a letter symbol to label a particular set, and once so identified, refer to this set by its label. Thus, if we agree to label the set of states of the United States by the letter A, then we can write

A = {states in the United States}.

Thereafter, the set of states in the United States may be referred to simply as A. Conventionally, capital letters are used for this purpose.

We have mentioned that a class roster is a means of specifying a particular set. Note that a child's name is not listed more than once in specifying the set. Once he is listed, he is designated as a member of the set. By the same token, {d, e, r} is the set of all letters in the word "deer" as well as in the word "red".

PROBLEMS

3. Using a common property, describe the set specified by
   a. {Alaska, Hawaii, Washington, Oregon, California}
   b. {Maine, Vermont, New Hampshire, Massachusetts, Rhode Island, Connecticut}
   c. {red, yellow, blue}

EQUAL SETS

When we write

A = {states in the United States}

we mean that A and {states in the United States} are symbols or names for the same thing. Whenever we use the equal sign "=" as in

$$5 + 2 = 7$$

we shall mean that the two sets of symbols are names for the same thing; in this case "5 + 2" and "7" are both names for the same number.

Note that {the first 5 letters of the English alphabet} is identical with {a, b, c, d, e}. To indicate that we have one and the same set, we say that these are equal sets and we write

{the first 5 letters of the English alphabet} = {a, b, c, d, e}.

In other words, if A is a set and B is a set, then

$$A = B$$

if both sets have exactly the same members.

Since the set consisting of the members Rosa, Eddy, and Leon is identical with the set consisting of the members Eddy, Rosa, and Leon, we can write

{Rosa, Eddy, Leon} = {Eddy, Rosa, Leon}.

Note that the order in listing the elements of a set is immaterial in specifying the set. The same set is specified by two different listings of the same members.

PROBLEMS

4. Are any of the following four sets equal?
   A = {1, 3, 5}
   B = {numerals representing the first three positive odd numbers}
   C = {135}
   D = {9}
   E = {the digits in the numeral 1351}

SUBSETS

A set is a collection of elements. The selection of certain elements from a given set will form a set. For example, from

$$A = \{a, b, c, d, e\}$$

we may form a set consisting of the elements, a, c, d:

$$B = \{a, c, d\}.$$

We say that B is a subset of A. Set B is said to be a subset of a set, A, if each element of B is also an element of A. Thus,

{Rosa, Eddy} is a subset of {Rosa, Eddy, Leon}

because each member of {Rosa, Eddy} is a member of {Rosa, Eddy, Leon}. However,

{Rosa, Anthony} is not a subset of {Rosa, Eddy, Leon},

because Anthony is not a member of {Rosa, Eddy, Leon}.

Observe that if

$$A = \{a, b, c, d, e\} \text{ and } B = \{b, e, c, a, d\}$$

then it is true that every element of B is an element of A (remember the order of listing of the elements is immaterial); so B is a subset of A. Since $A = B$, this example illustrates that one of the subsets that may be formed from a given set is simply the given set. This may be so taken for granted that the need to make such a statement is not at all apparent. However, this fact will have some undertones for us, as for example, when we examine certain special cases for subtraction.

We have noted that if

$$A = \{a, b, c, d, e\} \text{ and } B = \{b, e, c, a, d\}$$

then B is a subset of A; it is equally true that A is a subset of B. We can also see that

IF A IS A SUBSET OF B AND IF B
IS A SUBSET OF A, THEN A = B.

PROBLEMS

5. Which expression states that the letter y is an element of the set of letters in the word "Friday"?
   a. y is an element of {Friday}
   b. {y} is an element of {Friday}
   c. y is an element of {F, r, i, d, a, y}
   d. {y} is an element of {F, r, i, d, a, y}

6. {Tom, Harry} is a subset of {Tom, Dick, Harry}. Name six different subsets of {Tom, Dick, Harry}.

A SET WITH ONE MEMBER AND THE EMPTY SET

The set of all vowels in the word "cat" is a set with just one member, *a*. That is to say,

{the vowels in the word "cat"} = {a}.

This is an example of a set with a single member. It may conflict with our intuitive sense to think of a set with a single member since, in ordinary language, the word "set" connotes more than one object in the collection. An even more bizarre set that we shall describe is the set that has no members. Both of these mathematical concepts--of a set with one member and of the set with no members--are convenient ones; moreover, a vital question of logic requires the existence of such entities.

Logically, unless the concept of a one-member set is considered, it would make no sense to come up with "a" as the set of all vowels in the word "cat"; the letter "a" does not answer the question: "What is the *set* of all vowels in the word 'cat'?" Likewise, the same question of logic may enter into the consideration of the set with no members. A plea may be made that the question itself needs to be reworded. Instead of asking, "What is the set of vowels in the word 'cat'?", it may be more appropriate to ask, "What is the vowel in the word 'cat'?" This may sound sensible, but it does require a prior knowledge of the answer. Quite often, we do not know how many solutions we may have to a problem. With the understanding that there may be one, more than one, or no members in a set, there would be no need to rephrase the question each time a special situation is encountered. For example, the question, "What is the set of boys enrolled in this school?" might be equally applicable to the Yale, Columbia, and Vassar populations--or to one in which just one boy happens to be enrolled.

In thinking about a set with one member, there is a strong inclination to think of the set and the member that constitutes the set as one and the same thing and it is important to distinguish between the two. A case in point might be given, for example, in the cataloguing of books in the school library. Under the category of classics might be just the one book, *Treasure Island*. By itself, the book is not the same as the set of classics. If another book is added to the collection, the set of classics has changed; *Treasure Island* has not changed.

The empty set is the set with no members. Thus, the set of all boys enrolled in Vassar is an example of the empty set. The set of all months having nine Sundays is another example of the empty set. There may be many ways of illustrating the empty set but any example of the empty set has the same members as any other example of it because none of them has any members. This is why we say the empty set; there is only one such set. A notation for the empty set is { }. The empty space between the braces indicates that there are no members in the set. Another notation that is used for the empty set is the symbol ∅. With the first way of denoting the empty set the question may arise as to whether we had forgotten to list the elements within the braces. With the symbol ∅ this problem does not arise.

Recall that B is said to be a subset of A if each member of B is also a member of A. Another way to say this is

B IS A SUBSET OF A IF THERE IS NO MEMBER
OF B WHICH IS NOT ALSO A MEMBER OF A.

Both statements say exactly the same thing. As a consequence of the second statement, the empty set is a subset of

A = {Rosa, Eddy, Leon, Anthony}.

There is no member of { } that is not also a member of A. The empty set has no members. Thus, the empty set is a subset of every set.

PROBLEMS

7. Which of the following are equal sets?
   A = {a, e, i, o, u}
   B = { }
   C = {Monday}
   D = {women who are 20 feet tall}
   E = {the vowels in the English alphabet}
   F = {the days of the week}

8. Which of the sets in the above list is a subset of another in the list?

APPLICATIONS TO TEACHING

While both symbols, ∅ and { }, have been used here to denote the empty set, it is best to avoid introducing too many symbols simultaneously. Since the braces have been used for sets consisting of many members as well as for sets with one member we have kept to the use of the braces, { }, for the empty set for students in the primary grades. This notation does have the advantage of suggesting no members in the set.

As indicated above, the set with one member and the empty set may not seem to be easy concepts to present. Many teachers have, however, reported that children have been able to grasp these concepts quite easily. Since these sets will ultimately be associated with the numbers 1 and 0, they need to be included in our experiences with sets. There should be emphasis on the use of the article "the" in referring to the empty set. As in many other instances, for this level, the emphasis is largely by precept and example on the part of the teacher; there needs to be constant awareness of the proper use of language.

Both the proper use of language and the deliberate stress on certain critical terms are particularly important in view of the listening habits of some children. Some may not be able to grasp all that is said in long expressions. Some will attend to only part of what is said. Thus, aside from the cavalier reference to "the empty set" as "an empty set", there may be confusion between the words "set" and "subset". Unless conscious effort is made in enunciation, these terms may sound alike to the youngsters. In addition to marked effort in the proper use of language, constant and natural use of new terms throughout the day as occasions may arise has been found to be helpful. For example, there may be many instances of subsets that can be pointed out; during play period,

"a subset of the class that is on team A",

during reading,

"a subset of the ducklings in the pond",

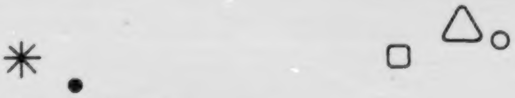
during music,

"a subset of the class that is playing the piano",

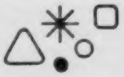
and so on.

Team membership offers excellent reinforcement of the fact that rearranging the members does not change the set. This is not an easy concept to teach and a variety of experiences may need to be provided leading to this notion. Some children are quite convinced that each time there is a new arrangement of the same members, a new set is formed. By way of illustration we might mention that the same members make up the set (team) regardless of who is on first base, who is on second and so on. If there is any change in membership, a different team is actually formed. Another illustration may be given in changing seat assignment in the classroom; the same set (class) of students is in each arrangement. Books may be arranged differently on a shelf. If there is no change in membership, each arrangement gives us the same set of books. The students will be faced with this concept again and again. For example, when we begin to compare sets of objects or when we partition a set into subsets to arrive at the concept of division, there will be opportunity to reinforce this notion.

We want to communicate the concept that a set is defined by the members; it does not matter how widely spaced these members may be. For example,



is the same set of objects as



Again, the illustration of team membership may be helpful. Initially, we may "choose sides" by grouping the members of each team together. Once the team membership is determined, the same members constitute the same team regardless of location of the individuals. The set of classics belonging to the school library, for example, may be defined by the cataloguing. Some of these books may be clustered in groups for a display; some may be on various shelves; some may be out on loan; spatial arrangement is immaterial to defining the set.

Our goal, through this discussion, has been to emphasize that set membership is independent of spatial arrangement. However, we recognize the intuitive aspects in visual perception. In a visual display, the spatial arrangement of a set of objects may suggest a natural grouping. Thus the arrangement

x x x x x   x x x x x   x x x x x
x x x x x   x x x x x   x x x x x

might suggest 3 groups of ten objects. Later on, when we examine the basis underlying our numeration system, we do capitalize on this tendency to group on the basis of spatial arrangement. For example, to arrive at a particular decimal numeral, a set of objects may be spatially grouped into subsets of 10's and 1's, and so on.

As was mentioned in the introduction, we shall ultimately elicit the concept of numbers from sets of objects. In particular, we shall associate a set with one member the number 1. While it is true that a set with one member cannot be considered (from the standpoint of logic) before a person has a concept of the number 1, nonetheless, it will be found that almost universally the children will already have "one" in their vocabulary. The word "single" may cause difficulty for some children. From a teaching standpoint, we may rely on using the words "one" and "a single" interchangeably to communicate some needed concepts. Again, classes react differently to the situation. Some teachers report success because the word "single" has been foreign to the students' vocabulary.

Along with emphasizing the natural use of language, we would like to emphasize the natural presentation of topics. By this, we mean a de-emphasis on decree: that is we do not wish to say these are the things you must learn and this is the way you learn them! By natural presentation, we also mean minimizing forced-feeding. At times, it may appear that teaching certain concepts reaches an impasse. Subsequently, in conjunction with presentation of some different topic, some student's remarks may reveal that what had appeared to be an impasse before is no longer one. It is likely that some incidental learning has occurred. It is also likely that there is reinforcement with other disciplines that, together with the presentation in mathematics, help to bring the concepts into focus. There is little need to insist on complete mastery immediately. Oftentimes, it is best to proceed with other developments when an impasse is apparent and return to the topic sometime in the future.

QUESTION

"How is it that the empty set is a subset of every set?"

A subset is a relative concept in the sense that it must be considered in relation to a given set. If every member (element) of a set B is also an element of a given set A, then B is a subset of A.

Suppose A = {house, toy, animal}

and B = {house, toy} .

Since each element of B--namely, house, toy--is an element of A, B qualifies to be a subset of A.

To say that each element of B is an element of A, is logically equivalent to say that there is no element of B that is not an element of A. It is by this second rephrasing that we can see more clearly that the empty set is a subset of every set. Compare the empty set with A = {house, toy, animal}. Is it true that there is no member of $\emptyset$ that is not a member of A? Certainly. Compare $\emptyset$ with B = {house, toy}. Is it true that there is no member of $\emptyset$ that is not a member of B? Clearly, we can apply this criterion comparing $\emptyset$ with any set and arrive at the same conclusion. Therefore, the empty set is a subset of every set.

To illustrate, we may consider the question, "What is the subset of this class whose members wish to fail this course?" If there are no members, then this particular subset is the empty set.

Another difficulty arises in connection with thinking of a set with a single member. Since a set is said to be a collection, the question is whether one can consider a single object a collection. If we think of "all objects that meet such and such conditions" as an alternate way of determining set membership, then the set of all vowels in the word "red" consists merely of the letter "e". {e} is the single member set, {all vowels in the word "red"}.

VOCABULARY

Elements of a Set*
Empty Set*
Equal Sets*
Improper Subset*
Member of a Set*
Set*
Subset*

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*The asterisk indicates that the term or phrase also appears in the glossary at the end of the book.

EXERCISES - CHAPTER 1

1. List the elements of each of the following sets whose descriptions are:
   - a. {the days of the week whose names begin with the letter W};
   - b. {positions on a baseball team};
   - c. {months of the year whose names have less than six letters};
   - d. {whole numbers between 7 and 8};
   - e. {the age on the nearest birthday for the students in your classroom};
   - f. {the capitals of Japan and England};
   - g. {the colors of the rainbow}.

2. Write a description of the set:
   - a. {Alaska, Hawaii};
   - b. {snips, snails, puppy-dog tails}.

3. Describe the common property of the elements of {cat, lion, tiger}.

4. Which of the following pairs of sets are equal?
   - a. {17} and {71}
   - b. {letters in the word bundle} and {n, d, b, l, u, e}
   - c. {•, q, ∅} and {q, •, ∅}
   - d. {zero} and {peacocks native to the North Pole}
   - e. {1, 2, 3, 4} and {a, b, c, d}
   - f. {are} and {era}
   - g. {M, i, s, p} and {the letters in the word "Mississippi"}

5. For each of the following, decide whether the statement is true or false and why.
   - a. 3 is a subset of {1, 2, 3}.
   - b. {ego} is a subset of {ego, je, I}.
   - c. It is possible for a set to be equal to one of its subsets.
   - d. {all birds in the world} is a subset of {all hens in the world}.

6. Given the set A = {rose, bee, tulip, beetle, dandelion}, write the subset of A described by:
   - a. {plants};
   - b. {insects};
   - c. {singers}.

SOLUTIONS FOR PROBLEMS

1. a. Erie, Huron, Michigan, Ontario, and Superior.
   b. Monday, Tuesday, Wednesday, Thursday, Friday, Saturday, and Sunday.
   c. This set is not well-defined. The set of objects depends on which Elsie. Even if Elsie is uniquely identified, it will be agreed that the set of objects changes in time. In order to specify the set, it is necessary to consider a particular Elsie at a particular moment.

2. b, c, and e are members of the set. a and d are not members of the set because a carrot is not an animal, nor is a tree.

3. a. {all states of the United States which border on the Pacific Ocean}
   b. {the New England states}
   c. {the primary colors}

4. Sets A, B, and E are equal. No others are equal. C is a single member set whose element is the numeral for one hundred thirty-five. The fact that no commas separate the digits, makes it different from 1, 3, 5. D contains the numeral for nine. Even though the sum of 1, 3, and 5 is 9, 9 itself is not another name for 1, 3, 5. D would be equal to {1 + 3 + 5} and 1 + 3 + 5 would be the single element of the set. The reason E is the same as A and B is that the digits of the numeral 1351 are 1, 3, 5 and 1. Recall, however, that an element is not repeated in a set, so the 1 should only be mentioned once.

5. c. only. a. is incorrect because {Friday} is a single member set whose element is the name of the fifth day of the week, which is not y. b. and d. are incorrect because {y} is a set. Neither {Friday} nor {F, r, i, d, a, y} have any members which are themselves sets.

6. {Tom}, {Dick}, {Harry}, {Tom, Dick}, {Dick, Harry}, {Tom, Dick, Harry}.

7. A = E; B = D

8. C and B are subsets of F.

CHAPTER 2

COMPARING SETS

BASIS OF COMPARISON

One of the ways that we have used to specify a set is to describe it by the property that the elements have in common. This method of classifying things can be extended to help distinguish one kind of set from another. Associated with this is the question: "What characteristic does one set have in common with another set?" Essentially, this is a classification problem that is one step removed from identifying the common property of elements within the set. For example, while the sets

> A = {lion, tiger, leopard} and C = {elephant, deer, cow, horse} are not equal, both of these are sets of animals, and may be distinguished from B = {house, tree, salt, rock}.

A further distinction might be that A is a set of carnivorous animals and C is a set of herbivorous animals. The point is that sets may be compared with another.

ONE-TO-ONE CORRESPONDENCE

One way of comparing two sets is by an element-by-element pairing. That is, an element of one set is paired with an element of the other set. To indicate a pairing we shall draw a double-headed arrow between the two members. Thus

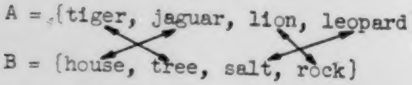
A = {tiger, jaguar, lion, leopard}

B = {house, tree, salt, rock}

shows that

lion	is paired with	salt;
jaguar	is paired with	tree;
tiger	is paired with	house;
leopard	is paired with	rock.

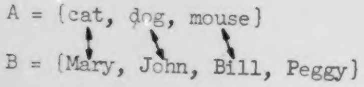
Another illustration of a pairing may be given by


A = {tiger, jaguar, lion, leopard}
B = {house, tree, salt, rock}

For our purpose, the concern is not so much that "lion" is paired with "rock" as that one member of A is paired with one member of B. Notice that in pairing the elements of B with those of A, each element of B is paired with an element of A and each element of A is paired with an element of B. When this happens, then we say that the sets match; also, we say that we have a one-to-one correspondence between the elements of the two sets. It can be seen that whether we can get a one-to-one correspondence between the elements of two sets does not depend on which element of B is paired with which element of A. For example, the pairings may be established by either of the diagrams above. In the first diagram it is easier to see at a glance that the pairing is a one-to-one correspondence than when the arrows are crossed.

ORDERING SETS

In pairing the elements of A with those of B (shown below), there is a member of B which is not paired with any element of A. This will be so regardless of how the elements are paired. In this case, we say that B has more members than A



We can also say that A has fewer members than B. Thus we can compare sets according to three possible outcomes:

A matches B;
A has more members than B;
A has fewer members than B.

Furthermore, all this can be accomplished without counting. Suppose C is the set of all children in the school and S is the set of seats in the school auditorium. By pairing, we can determine without counting whether one set has more members than the [illegible], one set has fewer members than the other, or the sets match.

PROBLEMS *

1. Which of the following pairs of sets match? For those that do not match, state which set has more members.
   a. {letters in the word "group"} and {g, o, p, r, u}
   b. {23} and {232}
   c. A = {1, 2, 3, 4, 5} and B = {c, d, e, f}
   d. B = {c, d, e, f} and C = {oyster, walrus, carpenter}
   e. A = {1, 2, 3, 4, 5} and C = { oyster, walrus, carpenter}

2. State why we do not necessarily have a one-to-one correspondence between the children in your class and their first names.

3. Show two different one-to-one correspondences between elements of the following pairs of sets.
   a. A = {animal, vegetable, mineral} and
      B = {carrot, plutonium, hippopotamus}
   b. A = {animal, vegetable, mineral } and
      C = {carrot, plutonium, beets}
   c. A = {animal, vegetable, mineral} and
      D = {iron, giraffe, turnip}

In one of the above problems we considered three sets, A, B, C, where

A = {1, 2, 3, 4, 5}
B = {c, d, e, f} and
C = {oyster, walrus, carpenter}.

Note that A has more members than B and that B has more members than C. Moreover, it can be seen that A has more members than C. This illustrates an important property called the transitive property. This property is important because it provides us with some means of working with numbers later. The property may be stated in general terms as follows:

IF A HAS MORE MEMBERS THAN B,
AND IF B HAS MORE MEMBERS THAN C,
THEN A HAS MORE MEMBERS THAN C.

---

* Solutions for problems in this chapter are on page 39.

This property is derived without recourse to counting. The conclusion sanctioned by this property gives us the comparison of A and C with a set, B, acting as intermediary. (In a sense, it tells us how A compares with C using B as a "yardstick".) Clearly, a transitive property is similarly applicable when A has fewer members than B, and B has fewer members than C. That is,

IF A HAS FEWER MEMBERS THAN B AND
IF B HAS FEWER MEMBERS THAN C, THEN
A HAS FEWER MEMBERS THAN C.

Furthermore, a similar property holds when the sets match, as we shall show later.

Observe that if A has more members than B, and if C has more members than B, no general conclusion can be made.

For example, if A = {1, 2, 3, 4, 5}, B = {c, d, e}, and C = {oyster, walrus, carpenter, cabbage}, then A has more members than B, C has more members than B, and A has more members than C.

If A = {1,2,3,4,5}, B = {c, d, e}, and C = {oyster, walrus, carpenter, cabbage, king}, then A has more members than B, C has more members than B, and A matches C.

If A = {1, 2, 3, 4, 5}, B = {c, d, e}, and C = {oyster, walrus, carpenter, cabbage, king, owl, pussy-cat}, then A has more members than B, C has more-members than B, and A has fewer members than C.

Thus, if A has more members than B, and C has more members than B, it is possible for A to have more members than C, to have fewer members than C, or to match C. So in this case we cannot determine the order of A and C.

By the transitive property, we have a way of ordering sets that do not match. If A = {1, 2, 3}, B = {a, b, c, d, e, f}, and C = {2, 5, b, ★ }, then as A has fewer members than C and C has fewer members than B, we can conclude that A has fewer members than B. Since A has fewer members than B we might order these sets: A, C, B. If D = {carrots, cabbage, carpenter, carousel, castenet}, we see that C has fewer members than D and D has fewer members than B. Here, by repeated comparison, D would be ordered between C and B. Thus we might order these sets A, C, D, B. Of course, the sets may be ordered equally well by the "more than" relation. For our purpose, ordering by the "fewer than" relation will lead directly to the ordering of numbers according to increasing size.

EQUIVALENT SETS

One of the possible outcomes from the pairing of the elements of two sets is that the sets match. If each element of A is paired with exactly one element of B and no element of B is left unpaired, we say that A matches B. Another way of describing this is that the elements of the sets are in one-to-one correspondence. A third way of saying this is that

A IS EQUIVALENT TO B.

The equivalence relation is transitive. If D= {1, 2, 3, 4, 5}, L = {c, d, e, f, g}, and W = {oyster, walrus, carpenter, cabbage, king}, then D is equivalent to L, L is equivalent to W, and D is equivalent to W. We can say this in general for any three sets A, B, and C:

(a) IF A IS EQUIVALENT TO B AND B IS
EQUIVALENT TO C, THEN A IS EQUIVALENT
TO C.

Let us consider the following sets:

A = {1, 2, 3, 4, 5}
B = {a, b, c}
C = { α, β, γ, δ, ϵ }
D = {△, ☆, 0}.

We see the following :

(1) A has more members than B
(2) C is equivalent to A
(3) C has more members than B
(4) D is equivalent to B
(5) C has more members than D.

In general, we can say for any four sets A, B, C, and D

IF A HAS MORE MEMBERS THAN B, AND
IF C IS EQUIVALENT TO A AND D IS
EQUIVALENT TO B, THEN C HAS MORE
MEMBERS THAN D.

A similar statement may be made in connection with the "fewer than" relation. That is,

IF A HAS FEWER MEMBERS THAN B, AND
IF C IS EQUIVALENT TO A AND D IS
EQUIVALENT TO B, THEN C HAS FEWER
MEMBERS THAN D.

In this manner, all sets may be ordered; sets that are equivalent belong in the same place in the order.

For sets that are equivalent, there are two additional properties that are of particular interest to us. If A = {1, 2, 3, 4, 5} and B = {a, b, c, d, e}, then A is equivalent to B. It is equally true that B is equivalent to A. We can see that by our pairing process, it must be true in general that

(b) IF A IS EQUIVALENT TO B, THEN
B IS EQUIVALENT TO A.

This is a property that the "more than" relation does not have. That is to say, if A has more members than B, then it is not true that B has more members than A. Neither does the "fewer than" relation have this property.

If A = {1, 2, 3, 4 5} and B = {3, 1, 4, 2, 5}, then certainly A is equivalent to B. In fact, here, A = B. Recalling that by A = B, we mean that both A and B represent the same thing (they are names for the same thing), it is clear that a set is equivalent to itself; that is,

(c) A IS EQUIVALENT TO A.

This is another property that the non-equivalent relations do not have. It is not true that A has more members than A; nor is it true that A has fewer members than A.

On the surface, the statement that a set is equivalent to itself may seem rather trivial. This is another of those statements that will have some repercussions later when we deal with numbers. It is not any more trivial than to assert that

$$2 + 5 = 7 \text{ because } 7 = 7.$$

Moreover, as was pointed out before, the last two properties stated for equivalent sets do distinguish equivalence from non-equivalence.

Equivalence relations possess all three of the properties mentioned, which have been identified by the letters (a), (b), and (c).

PROBLEMS

4. Write the order of the following sets, beginning with the set that has the fewest numbers.
   a. A = {letters of the alphabet in the word "peacock"}
      B = {letters of the alphabet in the word "letters"}
      C = {letters of the alphabet in the word "Mississippi"}
      D = {letters of the alphabet in the word "mathematics"}
   b. A = {1, 2, 3, 4}; B = {2, 3, 5, 7, 11, 13};
      C = {a, b, c, d, e, f}; D = { }.

5. Show how the transitive property may be applied to the following sets.
   A = {lion, tiger, leopard, elephant, mouse, cat}
   B = {house, tree, salt, rock}
   C = {the days of the week}

6. If A, B, C are the sets defined in Problem 5, and D, E, F are the sets so that
   D is equivalent to A
   E is equivalent to B
   F is equivalent to C,
   what is the order of D, E, F?

7. If A has more members than B, and C has more members than A, which set has the most members?

8. If B has more members than A, and C has more members than A, which set has the least members?

APPLICATIONS TO TEACHING

By the pairings that we have stated above, one member of a set is paired with exactly one member of a second set. Thus, it may be possible that we cannot completely pair the members. If A has fewer members than B, there will be at least one member of B that will be left unpaired. Furthermore, in a pairing, no more than one element of A is paired with a particular element of B. So, if A has more elements than B, there will be at least one element of A that will be left unpaired.

If no element of either A or B is left unpaired, then we have a one-to-one correspondence (abbreviated: 1-1 correspondence).

There are many many-to-one correspondences. For example, the rounding-off process is many-to-one. In rounding-off whole numbers to the nearest tens, the numbers

35, 36, 37, 38, 39, 40, 41, 42, 43, 44

are either "rounded-up" or "rounded-down" to 40. So this is a ten-to-one correspondence.

Our main concern here is with 1-1 correspondence. We use 1-1 correspondence for comparing sets according to how many elements they have. This in turn gives us a basis for comparing numbers. Aside from this, there will be many occasions in the mathematical career of the students, in which 1-1 correspondence will occur.

The students have been accustomed to thinking about pairs of objects that are alike, for example, pairs of mittens, pairs of shoes, pairs of socks, and so on. In our examples we have avoided the use of the word "pair" in this context because we do not want this restriction to get in the way of the concepts associated with numbers and with counting. In our development, we start with pre-number concepts that do not require the knowledge of numbers. From these concepts we derive the concepts of numbers. Our concentration on set-comparison by equivalence is to prepare for the concept that if sets are equivalent, then they generate the same number.

The word "equivalent" may cause difficulty for some children. However, this may be again a matter of individual reaction. Some teachers have found that some children apparently cope with this word successfully because the word is foreign to the children's vocabulary. The phrase "as many as" is also used in conjunction with developing the notion of equivalence. The words in this phrase are more easily handled, but the longer phrase demands more attention on the part of the children. Some children may attend to only part of the phrase. For example, in response to the request to produce a set with as many members as a given set, the child may merely produce one with many members.

The notion of separating objects into equivalent sets or classes also underlies our thinking of many names for a number. For example,

$$\frac{1}{2}, \frac{2}{4}, \frac{3}{6}, \frac{4}{8}, \frac{5}{10}, \ldots$$

all name the same number, and we can think of all these fractions as being in the same equivalence class. Any fraction in this class is equivalent to another and we may use any one fraction in this set as a representative of the set. Usually, we choose the fraction that is reduced to lowest terms as the representative and consider that this represents the number. But this is not always the case. For example, if we have the problem

$$\frac{1}{2} + \frac{2}{3}$$

neither the fraction $\frac{1}{2}$ nor $\frac{2}{3}$ are convenient representatives for the numbers that we have in mind. From the set of fractions for one-half

$$\{\frac{1}{2}, \frac{2}{4}, \frac{3}{6}, \frac{4}{8}, \frac{5}{10}, \ldots\}$$

and the set of fractions for two-thirds

$$\{\frac{2}{3}, \frac{4}{6}, \frac{6}{9}, \frac{8}{12}, \frac{10}{15}, \ldots\}$$

we pick the convenient ones with common denominators to work with in our problem. Thus

$$\frac{1}{2} + \frac{2}{3} = \frac{3}{6} + \frac{4}{6} \qquad \text{or} \qquad \frac{1}{2} + \frac{2}{3} = \frac{6}{12} + \frac{8}{12}$$

and so on. Out of these, the ones we consider to be the most convenient ones to use are the ones with the least common denominators.

QUESTION

"How does the transitive property give us a way of ordering sets that do not match?"

This is in reference to the transitive property of "more than" or "fewer than" (page 29). Taken out of context, the question would be inappropriate. For sets that match, a transitive property applies, but this does not give a way of ordering sets in the manner that we have in mind: according to the number of elements. By this criterion, one

4

set that matches another cannot be said to have a higher or lower order than the other.

If $A = \{a, b, c, d, e, f\}$
$B = \{\text{ace, king, queen, jack}\}$
$C = \{\text{book, wagon}\}$
$D = \emptyset$
$E = \{a, b, c\}$,

then no two of the sets match. Comparing A with B, we see that B has fewer members than A. So, in increasing order of the number of elements, we have B, A. Comparing C with B, we see that C has fewer elements than B. By the transitive property, C has fewer elements than B and B has fewer elements than A, means C has fewer elements than A. Thus, in increasing order, we have C, B, A, and similarly by repeating this process we can get the order

D, C, E, B, A.

VOCABULARY

As Many As (As Many Members As)*
Equivalent Sets*
Fewer Than (Fewer Members Than)*
Match*
More Than (More Members Than)*
One-to-One Correspondence*
Pairing*
Transitive Property

EXERCISES - CHAPTER 2

1. If the sets match, show a pairing. If they do not, tell which set has fewer members than the other.

   a. $A = \{\square, O, \triangle, \bigstar\}$
   $B = \{X, I, V, M\ C\}$

   b. $C = \{\text{cow, tree, blimp}\}$
   $D = \{\text{dirigible, trunk, milk}\}$

2. Order the sets X, Y, Z.

   $X = \{1, 2\}$ $\quad Y = \{3, 4, 5, 6\}$ $\quad Z = \{789\}$

3. Gloria is taller than Andrea, and Mary is taller than Gloria. Can the concept of transitivity be applied here? If not, why not? If so, what conclusion can be drawn?

4. In attempting to place the elements of P in 1-1 correspondence with the elements of Q, if we run out of members of P before we run out of elements of Q, what can be said of the relationship between P and Q?

5. The elements of which sets can be put in a 1-1 correspondence?
   a. A = {living human beings} B = {functioning human brains}
   b. C = {social security numbers} D = {income tax returns filed}
   c. E = {consonants in "I"} F = {women who have been president of the U. S.}
   d. G = {the human senses} H = {normal number of toes on a dog's hind foot}
6. Name three ways of describing the fact that A matches B.

SOLUTIONS FOR PROBLEMS

1. a. These sets match. In fact they are equal, so one "natural" pairing would be to pair each member with itself.
   b. These sets match. There is only one pairing since each is a single member set: {23}
   {232}
   c. A and B do not match. A = {1, 2, 3, 4, 5}
   B = {c, d, e, f}
   Since there is an element of A left over in any pairing, A has more members than B.
   d. B has more members than C.
   e. A has more members than C.
2. There may be more than one child having the same first name.
3. a. A = {animal, vegetable, mineral}
   B = {carrot, plutonium, hippopotamus}

   A = {animal, vegetable, mineral}
   B = {carrot, plutonium, hippopotamus}

   b. A = {animal, vegetable, mineral}
   C = {carrot, plutonium, beets}

   A = {animal, vegetable, mineral}
   C = {carrot, plutonium, beets}

   c. A = {animal, vegetable, mineral}
   D = {iron, giraffe, turnip}

   A = {animal, vegetable, mineral}
   D = {iron, giraffe, turnip}

4. a. A = {p, e, a, c, o, k}
B = {l, e, t, r, s}
C = {m, i, s, p}
D = {m, a, t, h, e, i, c, s}
The order requested is C, B, A, D

b. D, A, B or D, A, C. Since B and C are equivalent sets, they must occupy the same position in any ordering.

5. Since B has fewer members than A and A has fewer members than C, it must be true that B has fewer members than C.
or
C has more members than A; A has more members than B. Therefore, C has more members than B.

6. The increasing order of A, B, C is B, A, C; D, E and F must be ordered E, D, F because equivalent sets must occupy the same position in any ordering.

7. C. By transitivity, we can order the sets as B, A, C, starting with the set that has the fewest elements.

8. A. This is a case in which we cannot determine the order of B and C. We only know that both have more members than A.

Chapter 3

WHOLE NUMBERS

NUMBER PROPERTY OF SETS

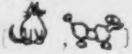
The concept of number is developed from the concept of sets. In Chapter 2 we compared sets on the basis of characteristics which they had in common. We also ordered sets. In this chapter we shall focus our attention on one of these common properties and develop the concept of number.

Recall that sets can be compared according to different criteria. A set of red balloons and a set of red blocks share the common characteristic of color. A set of blue blocks, a set of green blocks and a set of red blocks are each composed of elements which are blocks.

In the last chapter, attention was given to pairing the elements of two sets. If a one-to-one correspondence can be set up between the elements of two sets, they were said to be equivalent. For example, {Leon, Rosa, Eddy} is equivalent to {a, b, c} because their members can be paired with none left over. It is certainly possible to name many other sets which are equivalent to these; indeed, we could never exhaust all the possibilities. These sets share a common property; that is that they have the same number of members.

Similarly the sets

A = {a, b}

B = {}

C = { ⌂, ○ }

D = {Don, Len}

are each equivalent to any other in this list. They share a common property of each having two elements.

Every set has this number property. We call this characteristic the number of the set.* It is determined by the number of elements in the set. Sets which are equivalent have the same number. To simplify the terminology, we denote the number property of a set A as N(A). We can rephrase the statement that equivalent sets have the same number by saying:

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*We shall call this the cardinal number of the set. Cardinal number will be discussed later.

IF THE SETS A AND B ARE EQUIVALENT,
THEN N(A) = N(B).

Note that this does not say A = B The statement A = B is only true if A and B have the same members.

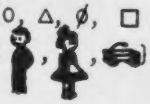
PROBLEMS*

1. Describe a property which the following two sets have in common with each other.

   D = {doll, balloon, tinker toy}
   W = {block, wagon}

2. Identify the number of the sets using the notation N( ).
   a. S = {b, d, f, h, j}
   b. P = {?, ., :, "}
   c. A = {letters in "abbreviation"}

3. Given:

   A = {r, e, a, d}
   B = {2, 4, 6, 8, 10 12, 14}
   C = {0, △, ∅, ☐, ✹, X}
   D = {  }

   Find
   a. N(A)
   b. N(B)
   c. N(C)
   d. N(D)

ORDERED SETS

Frequently, the elements of a set present themselves in a natural order. For instance, most English speaking people would list the members of the set of vowels as {a, e, i, o, u}. It is natural to list the elements in this order because this is the order in which they were learned. It is convenient because without undue checking one can be sure he has not omitted any member.

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*Solutions for problems in this chapter are on page 51.

Similarly, it is natural to list the members of the set of letters of the alphabet as

{a, b, c, d, e, f, g, h, i, j, k, l, m, n, o, p, q, r, s, t, u,
v, w, x, y, z}

In ordinary writing we write this set as

{a, b, c, ..., z}.

The three dots, ..., mean "and so on in the same manner". They are used to indicate the omission of certain members.

Essentially, to "order" things is to list or arrange them in some particular fashion. One can then say of each element which of the other elements it "precedes". We do this by comparing pairs of elements in the list and deciding which element precedes the other. The word "precedes" may be replaced by "above", "below", "shorter than", "greater than", and so on depending on the elements to be ordered.

For example, consider the set of names

{James, Wilson, Smith, Alton} .

If we order these elements alphabetically we have

{Alton, James, Smith, Wilson}.

We call this set an ordered set.

STANDARD SETS

Let us establish some ordered sets beginning with the set {1}. We continue

{1, 2},
{1, 2, 3},
{1, 2, 3, 4},

and so on.

We see that each of these sets is a subset of each of the following sets. Thus,

{1} is a subset of {1, 2},
{1, 2} is a subset of {1, 2, 3},

and so on.

By comparing these sets, called standard sets, we can determine which belongs before the others in ordering these sets. For example, we see immediately that {1, 2, 3} belongs before {1, 2, 3, 4, 5, 6} in ordering these standard sets.

PROBLEMS

4. For each of the following sets, state whether the elements are apparently ordered; if an order is apparent, describe what might be the determination of the order.
   a. {1, 2, 3, 4, 5}
   b. {5, 4, 3, 2, 1}
   c. {accordion, albatross, brain, bubble, gum, humbug}
   d. {student, teacher, principal, superintendent}
   e. {father, son, mother, daughter}
   f. {father, mother, son, daughter}
   g. {q, w, e, r, t, y, u, i, o, p}
   h. {5, 4, 1, 3, 2}

5. Given an ordered set,

   H = {thumb, index, middle, ring, pinky},

   a. Show the 1-1 correspondence between the elements of H and a standard set.
   b. If S = {Dorothy, Rosie, Laurie, Nancy, Susan} give a subset of H that is equivalent to S; what is N(S)?
   c. Describe how counting on our fingers implies finding a set that is equivalent to a standard set of number names.

CARDINALITY AND ORDINALITY

Let us consider the sets

A = {Deane, Leo}
B = {Don, Len}
C = {Ruth, Margaret}
D = {Elaine, Mabel}.

Each of these sets are equivalent to any other, since the elements of any two of the sets can be put into 1-1 correspondence. Let us consider all the sets equivalent to any one of these given sets, for example {Deane, Leo}. Among the sets equivalent to this set is the standard set {1, 2}. These sets all possess a common property: their equivalence to the standard set {1, 2}. This property is independent of the elements of the set. We call this common property the number two. We say the number property of the set A = {Deane, Leo} is 2. We write this N(A) = 2. This number property of a set is the cardinal number or cardinality of the set, and the number itself, a cardinal number.

Similarly the number property of {1} is 1; of {1, 2, 3} is 3; and so on. Notice that the number property of any standard set is the number named by the last element in the set. The empty set is assigned the cardinal number zero. That is N(∅) = 0. The words "one", "eight", "ninety-nine", and so on, are names of cardinal numbers. This concept can be considered entirely separately from the phenomenon of order.

Much has been said about the ordering of sets and of elements within sets. In this reference, the words first and last have been used. The fact that we can talk about the third letter of the alphabet or the fiftieth state of the Union, depends on the ordinality of numbers. The words first, second, thirty-eight, and so on are names of ordinal numbers. These are independent of quantity and can only be considered relative to some frame of reference. That is, we cannot speak of the third quarter in a football game without implying that there were a first and a second quarter. However, the third quarter only refers to one of the implied three quarters. Both aspects of number are contained in the statement: Jimmy is the third child of our seven children. Note that an ordinal number requires a set of at least the corresponding cardinal number of members. Jimmy is the third child requires at least a set of three. On the other hand, a cardinal number does not necessitate ordinality of its members. The number two is the property of {chicken, egg}; the question of the ordinality of the members of this set has occupied minds for years!

At this point, we want to remark on the common usage of language with reference to the ordinality and cardinality of numbers. Quite often, as in the case of "Page 3", a number is meant to be used in an ordinal sense even though it is stated as a cardinal number. The identification "Page 3" refers to the third of a series of pages rather than to three pages.

PROBLEMS

6. Identify each number in the following as to whether the use is ordinal or cardinal.
   a. There are 3 blocks on the table.
   b. John is number 5 in line.
   c. My address is 164 State Street.
   d. Seventeen children are in this class.
   e. Joyce read Chapter 7 last night.

7. Identify each number in the following statements as to whether the use is ordinal or cardinal.

   "Two of you in Group Three will have the assignment of looking up the history of zero in the encyclopedia. You will find that of all these 30 volumes, Volume 17, Part 2 contains material that will be most helpful to either one of you."

FINITE AND INFINITE SETS

The set of cardinal numbers, when arranged in order, is endless. Given any standard set, it is always possible to find another set with larger cardinality. We say that the set of cardinal numbers is infinite.

Any nonempty set, A, which is equivalent to a standard set is called a finite set. In other words, if a set A is a finite set, its elements can be counted, and such a counting would come to an end.

Examples of finite sets are

P = {a, b, c, ..., x, y, z}
Q = {children in this class}
R = {houses on Main street}.

Examples of infinite sets are

S = {cardinal numbers} = {0, 1, 2, ...}
T = {even cardinal numbers} = {0, 2, 4, ...}.

ORDER OF NUMBERS

The numbers named by the set of numerals

{0, 1, 2, ...}

are called the whole numbers. As in the case of

{a, b, c, ..., x, y, z}

we have used the three dots to indicate the omission of certain elements. The difference in the use of the three dots in

$$\{0, 1, 2, \ldots\}$$

is that no end is indicated in the list of whole numbers. The set of whole numbers is an infinite set.

If zero is omitted from the set

$$\{0, 1, 2, 3, \ldots\}$$

we have the set of counting numbers or natural numbers. Thus the set of counting numbers is

$$\{1, 2, \ldots\}.$$

Whole numbers can be ordered by means of standard sets. Two sets such as

$$A = \{a, b, c, d, e\}$$

and

B = {△, ♡, □, ⧗, ⊗}

are equivalent. Each of these sets is equivalent to the standard set

$$\{1, 2, 3, 4, 5\}.$$

Hence the cardinal number of these sets is 5.

If a standard set S has fewer members than a standard set P, then the cardinal number of S is defined to be less than the cardinal number of P. For example $\{1, 2, 3\}$ has fewer members than $\{1, 2, 3, 4, 5\}$ and hence 3 is less than 5. We write this

$$3 < 5.$$

The symbol "<" means "is less than".

When the elements of the set of whole numbers is written in order, 0, 1, 2, ..., each number is less than any number that succeeds it in the sequence. Thus

$$0 < 1 < 2 < 3 < 4 \ldots$$

The statement $3 < 5$ may be written

$$5 > 3.$$

which is read "5 is greater than 3". The symbols < and > mean "is less than" and "is greater than" respectively.

If we choose any two whole numbers a and b, exactly one of the following statements is true:

$$a < b$$
$$a = b$$
$$a > b.$$

PROBLEMS

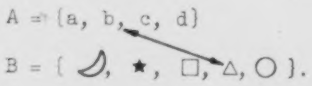
8. If $S = \{b, d, f, h, j\}$, $P = \{?, ., :, "\}$, and $A = \{a, b, r, e, v, i, t, o, n\}$, put the sets in increasing order and then order their numbers, using the symbol $<$.

9. Without knowing the numbers of two sets, say X and Y, what must be true of $N(X)$ and $N(Y)$?

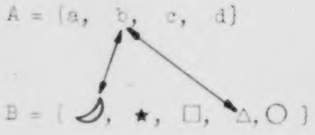
APPLICATIONS TO TEACHING

The focus on sets and other pre-number concepts provides a background for the concept of number introduced in this chapter. If there is a 1-1 correspondence between the elements of two sets, then they are said to be equivalent and have the same cardinal number or the same number property. To determine whether there is a 1-1 correspondence between the elements of two sets, an element of one set is paired with an element of the other set. The use of the word pair is non-mathematical. The two elements that are associated form a pair.

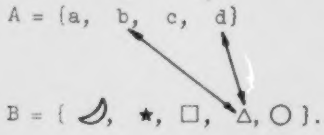
For our purpose, once a pair is so determined, neither of the elements in the pair is to be associated with any other element to form another pair. Thus, for A = {a, b, c, d} and B = {🍌, ★, □, △, ○}, if we decide to pair b with △, we have



If, in the attempt to get a 1-1 correspondence, b is paired with △ then b is not to be paired with any other member of B. Neither is △ to be paired with any other member of A. Thus we cannot seek a 1-1 correspondence between the members of A and B by



or



Of course, b may have been selected to be paired with ☽ at the outset. Then b is not to be paired with ★, □, △, or ○ ; nor is ☽ to be paired with a, c, or d. The one-to-many and many-to-one correspondences illustrated in these last two diagrams as well as many-to-many correspondence will be discussed in the next chapter.

In the children's books the pairings are indicated by connecting lines from one object to another much as we have used the arrows on these pages. Special attention may need to be devoted to explaining the meaning of these extra lines (dotted or otherwise) on the printed page as they may be a source of confusion. Having the children trace over these lines themselves may be helpful. Thus they are actively engaged in connecting or pairing the objects. For the same reason, active participation in pairing objects on the flannel board using yarn to define the pairings will be helpful. The pairings may result in exhausting the elements of one set without exhausting the elements of the other. When the elements of both sets are exhausted simultaneously, the sets match.

Pairing then refers to the elements and matching to the sets. When the sets match, they are said to be equivalent. Thus equivalence implies the same number of elements. Note that equal sets are always equivalent. Sets are equal only if they have the same members; therefore, the number of elements must be the same. However, it is not true that equivalent sets must necessarily be equal. $\{1, 2, 3, 4, 5\}$ and $\{1, 2, 3, 4, 6\}$ are equivalent but not equal; there is a 1-1 correspondence between the elements of these two sets.

Just as the determination as to whether the number properties of two sets are equal depends on whether the sets match, the order of two numbers depends on the result of set comparison. If A has fewer members than B, then $N(A) < N(B)$. The characteristics ascribed to numbers derive from characteristics observed for sets, not the other way around.

Occasionally, a child may understand 1-1 correspondence and the process of counting but still may be unsuccessful because he cannot keep track of what he has counted and what he has not counted. For such a child it may be necessary to actually suggest some systematic strategies in attacking the problem. For example, if the objects are in a horizontal row and he still skips around counting the objects in a random fashion, it might be suggested that he proceed from left to right as in reading.

In counting, it does not matter which element of a set is paired with a given element in the appropriate standard set. The same number property is obtained regardless of the pairings used. By contrast, in ordinal use of numbers, it is assumed that there is a pre-determined order in the given set as well as in the standard set. That is, the elements are ordered by associating with each element as the first, second, third element, and so on as the case may be. The ordinal numbers may not be in the vocabulary of some children. However, it has been observed that many children do know what these words mean. In such cases, apparently some incidental learning has occurred.

QUESTION

"What is the difference between 'equivalent sets' and 'equal sets'?"

If A and B denote sets, to say that A = B, we mean that A and B are both names for the same set. Thus, if A = {shoe, doll, wagon} and B = {shoe, wagon, doll}, we can say that A and B are equal: both A and B consist of the same members.

On the other hand, the requirement for sets to be equivalent is less demanding: if the sets match, then they are equivalent. Thus, if A = {shoe, doll, wagon} and C = {a, b, c}, then both sets have the same number property (both sets match). Hence, A is equivalent to C even though A is not equal to C. If A is equal to B, then A and B are necessarily equivalent. A and B have exactly the same members, therefore the number of members are necessarily equal. If A is equivalent to B, then A and B are not necessarily equal. Having the same number of members does not mean that these sets must therefore be identically constituted.

VOCABULARY

Cardinal Numbers*
Counting Numbers*
Finite Set*
Greater Than*
Infinite Set*
Less Than*

Natural Numbers*
Number Property of a Set*
Ordered Set
Ordinal Number
Standard Sets*
Whole Numbers*

EXERCISES - CHAPTER 3

1. Name the number property of the sets:
   a. {□, △, ○, ⬡}
   b. {243}
   c. {zero}
   d. {letters in the word "deeded"}
   e. {the number of vowels in "bureau"}
   f. {counting numbers less than 1}

2. Here are four sets: $A = \{a, b, c, d\}$
   $B = \{1, 2, 3\}$
   $C = \{\ \}$
   $D = \{\alpha, \beta, \gamma\}$

   Identify the number properties of these sets; write all the relationships you can, using the numbers and the symbols $<$, $=$, $>$. For example, $3 < 4$.

3. Suppose you want to explain "wide" to someone who speaks no English and you do not speak his language. How would you go about conveying to him the idea of "wide"?

4. M = {man, fish, ape, amoeba, lizard}
   Rewrite the elements of M in some more intuitively logical order and describe how it is determined.

5. List the elements of {9, 6, 11, 4, 3, 1, 10, 8, 7, 5, 2} in such a way that its number can be determined without counting.

SOLUTIONS FOR PROBLEMS

1. Answers may vary; e.g., they are both sets of objects which are children's toys.

2. a. $N(S) = 5$
   b. $N(P) = 4$
   c. Since $A = \{a, b, r, e, v, i, t, o, n\}$, $N(A) = 9$.

3. a. 4
   b. 7
   c. 6
   d. 3

4. a. The elements are listed in increasing order.
   b. The elements are listed in decreasing order.
   c. These words are in alphabetical order.
   d. This describes the positions in the ascending hierarchy in a school.

e. Males listed first in decreasing order of age and then females are listed in order of age.

f. Two apparent rules of order operate here also. The adults are listed before the children and male takes precedence over female.

g. No order is obvious, so "not apparent" would be correct. However, the letters happen to be in the order in which they appear on the third row of a standard typewriter.

h. Again, no order is apparent. If the set is renamed by the words {five, four, one, three, two}, you can see they are in alphabetical order.

5. a. H = {thumb, index, middle, ring, pinky}

$$\begin{array}{ccccccc} H = \{ & \text{thumb}, & \text{index}, & \text{middle}, & \text{ring}, & \text{pinky} & \} \\ & \updownarrow & \updownarrow & \updownarrow & \updownarrow & \updownarrow & \\ \{ & 1, & 2, & 3, & 4, & 5 & \} \end{array}$$

b. The subset of H which is equivalent to S is the improper subset H = {thumb, index, middle, ring, pinky}. N(S) = 5 as part a. shows.

c. As one counts, he is listing the elements of a standard set. When counting on our fingers, we usually touch one and say a number. This is pairing fingers with numbers so that the sets match. Hence we have found a set, a set of fingers, which is equivalent to {1, 2, 3}.

6. a. Cardinal

b. Ordinal

c. Ordinal

d. Cardinal

e. Ordinal

7. Cardinal numbers are: "two", "zero", "30", "one"; those used as ordinal numbers are: "Three", "17", "2".

8. The sets in increasing order are P, S, A . Since $N(S) = 5$, $N(P) = 4$ and $N(A) = 9$, the numbers must be ordered $4 < 5 < 9$.

9. Exactly one of the following statements must be true:
$N(X) < N(Y)$ or $N(X) = N(Y)$ or $N(X) > N(Y)$.

Chapter 4

SET OPERATIONS

SET UNION

Suppose we consider the following sets:

A = { △ , ○, □,}

B = { ⌂, ✣, ⧗ , 8 ,✳ }

Let us join the elements of sets A and B to form a new set. This new set consists of all the elements belonging to A or B or both and is called the union or join of A and B. We write

A ∪ B = { △, ○, □, ⌂, ✣, ⧗, 8 , ✳ }

and read "A ∪ B" as "A union B".

This process of joining two sets is called an operation on sets. Since we join just two sets at a time it is called a binary operation. For our present purpose, we use this operation of joining only if the two sets do not have any members in common. The elements △, ○, □, are members of A but not of B. It is equally true that none of the members of B is a member of A. If two sets do not have any members in common, as in this case, then we say that the sets are disjoint sets. For example, the set of boys in a classroom and the set of girls in the classroom are disjoint sets; the union of these two sets is the set of boys and girls in the classroom.

PROBLEMS*

1. Find the union of each pair of sets:
   - a. A = {1}; B = {2}
   - b. A = {1, 2}; B = {3}
   - c. A = {1, 2, 3}; B = { }
   - d. D = {1, 2, 3}; L = {a, b, c, d, e}
   - e. L = {a, b, c, d, e}; D = {1, 2, 3}

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*Solutions for problems in this chapter are on page 66.

f. $S = \{a, b, c, d, e, 1, 2, 3\}$; $G = \{\alpha, \beta, \gamma, \delta\}$

g. $D = \{1, 2, 3\}$; $G = \{\alpha, \beta, \gamma, \delta\}$

h. $L = \{a, b, c, d, e\}$; $A = \{1, 2, 3, \alpha, \beta, \gamma, \delta\}$

2. a. Which of the problems above illustrates the result of the union of a set with the empty set?

   b. If A is a set, what is $A \cup \{\ \}$?

3. If V = {aardvark, bear, cougar, deer, elephant, fox, giraffe, hyena, ibex, jackal, kangaroo, llama},

   W = {aardvark, cougar, fox, jackal},

   and X = {bear, deer, elephant, giraffe, hyena, ibex, kangaroo, llama}, what is $W \cup X$?

PROPERTIES UNDER UNION

There are a few properties under the union operation that will have important bearing for us when we work with numbers. If

B = {Anthony, Barry, Charles, Douglas}
and G = {Ethel, Florence, Grace},
then $B \cup G$ = {Anthony, Barry, Charles, Douglas, Ethel, Florence, Grace}
and $G \cup B$ = {Ethel, Florence, Grace, Anthony, Barry Charles, Douglas}.

Observe that $G \cup B$ has the same members as $B \cup G$, hence

$$G \cup B = B \cup G$$

In fact, it is always true that for any two sets it is immaterial whether the first set is joined to the second, or the second is joined to the first; the same set results by the union. To express this fact, we say that

THE OPERATION OF UNION OF SETS IS COMMUTATIVE.

In other words, if A and B are sets, the commutative property under the union operation states that

$$A \cup B = B \cup A.$$

Another way to describe this is: under the union operation, the order of joining does not matter.

There are many actual situations in which commutativity holds. These are instances of commutative operations. For example, it makes no difference in what order the left sock or the right sock is put on. The final result of applying the operation on both objects is identical in each case.

Taking three red marbles from a sack, four green marbles from a second sack, and putting these together into a third sack is another illustration of a commutative operation. Taking four green marbles from the second sack and three red marbles from the first to put together into the third sack would net the same result.

On the other hand, there are situations where the results do depend on the order in which the operation is carried out. For example, applying a coat of red paint on top of a coat of green paint gives a different visual effect than reversing this procedure. Therefore, it is pertinent to point out commutativity when it does occur.

Another property for sets that is of significance for numbers is known as the associative property under the union operation. The combination of more than two sets is involved in this property. Such a combination is possible because a set is specified by the union of two sets. With a set so formed, the union with still another set may be obtained. To illustrate, if $D = \{1, 2, 3\}$ and $L = \{a, b, c, d, e\}$, then

$$\begin{aligned} D \cup L &= \{1, 2, 3\} \cup \{a, b, c, d, e\} \\ &= \{1, 2, 3, a, b, c, d, e\}. \end{aligned}$$

If $G = \{\alpha, \beta, \gamma, \delta\}$, we can join G to $D \cup L$ to get

$$\begin{aligned} (D \cup L) \cup G &= \{1, 2, 3, a, b, c, d, e\} \cup \{\alpha, \beta, \gamma, \delta\} \\ &= \{1, 2, 3, a, b, c, d, e, \alpha, \beta, \gamma, \delta\}. \end{aligned}$$

The parentheses around $D \cup L$ shows that L is first joined to D. As $D \cup L$ is a set, it is possible to join another set to $D \cup L$, and this is indicated by the union with G. Thus new sets may be formed successively by this process. The possibility of creating new sets repeatedly is crucial to the property that we have in mind. However, the property further, and more directly, has to do with examining the result of the repeated joinings.

If D, L, and G are as above, we see that we can get the result of $(D \cup L) \cup G$. Let us now consider the union of L and G, and then join this set to D. The union of L and G is

$$\begin{aligned} L \cup G &= \{a, b, c, d, e\} \cup \{\alpha, \beta, \gamma, \delta\} \\ &= \{a, b, c, d, e, \alpha, \beta, \gamma, \delta\}. \end{aligned}$$

The union of D and this set is then

$$\begin{aligned} D \cup (L \cup G) &= \{1, 2, 3\} \cup \{a, b, c, d, e, \alpha, \beta, \gamma, \delta\} \\ &= \{1, 2, 3, a, b, c, d, e, \alpha, \beta, \gamma, \delta\}. \end{aligned}$$

Compared with

$$(D \cup L) \cup G = \{1, 2, 3, a, b, c, d, e, \alpha, \beta, \gamma, \delta\}$$

that we have above, it is clear that the same set results from the two procedures. In general,

FOR SETS A, B, AND C, IT IS TRUE
THAT $(A \cup B) \cup C = A \cup (B \cup C)$.

What is conveyed by this property is that B may be joined with either A or C first; the final result of the union of all three sets will be the same. That is to say, B may be associated first with A to form $A \cup B$ or B may be associated first with C to form $B \cup C$; the union of either of these with the remaining set (C or A as the case may be) is the same in both cases. This is what we mean when we say that the union of sets is an associative operation. In other words, in a union involving three sets, the different ways the sets are grouped to form unions in the intermediate stage does not affect the final result.

Because we have this option in grouping, $(A \cup B) \cup C$ and $A \cup (B \cup C)$ denote the same set. Consequently, we need not specify in the notation how the union is to be accomplished. Therefore, the notation may be simplified by dropping the parentheses and writing

$$A \cup B \cup C.$$

Since $A \cup B \cup C$ is a set, the union may be extended again and again. With the same kind of analysis, it is clear that the associative property under the union operation is equally applicable to more than three sets.

A third property that will be of interest to us is one illustrated by the example $\{1, 2, 3\} \cup \{\ \}$. As the union is composed of all the elements in each of the two sets, and since the empty set has no members, the union is precisely $\{1, 2, 3\}$. Therefore, we must have

$$\{1, 2, 3\} \cup \{\ \} = \{1, 2, 3\}.$$

In general, if A is a set, then it is true that

$$A \cup \{\ \} = A.$$

This is parallel to the situation in arithmetic when 0 is involved in addition, such as: $3 + 0 = 3$. In fact, it will be pointed out that the union with the empty set is the basic concept from which we derive the property of 0 under addition.

COMPLEMENT

Let $A = \{a, b, c, d, e, f\}$ and $B = \{b, e, d\}$

We see that B is a subset of A. The set of all the elements of A which are not elements of B is called the relative complement of B to A.

The relative complement of B to A is written $A - B$. Then

$$A - B = \{a, c, f\}.$$

We read this "the relative complement of B to A is the set {a, c, f}".

Sometimes we shorten "relative complement of B to A" to "the complement of B".

Notice that if $C = A - B$, then

$$B \cup C = A.$$

Notice if C is the complement of B, then B is the complement of C.

Many examples of complements abound in actual situations. For example, the set of boys in the classroom is the complement of the set of girls in the classroom. These are disjoint sets that together complete the set of boys and girls in the classroom. The set of vowels and the set of consonants might be another example of complementary sets. A complementary set will also be referred to as a remainder set. This concept will be developed further when we talk about subtraction.

PROBLEMS

4. In each of the following problems, state which property or properties are indicated.
   a. $\{a, b, c\} \cup \{d, e, f, g\} = \{d, e, f, g\} \cup \{a, b, c\}$
   b. $A \cup (B \cup C) = A \cup (C \cup B)$
   c. All the children in K-1 or the children in the second grade are children in the kindergarten or children in the first two grades.

5. Given: $A = \{1, 2, 3\}$, $B = \{4, 5\}$, $C = \{6, 7, 8\}$,

   $D = \{1, 2, 3, 4, 5\}$ and $E = \{4, 5, 6, 7, 8, 9\}$

   Find
   a. The relative complement of A to D.
   b. The relative complement of C to E.
   c. $(A \cup C) \cup B$
   d. $D \cup C$

6. If M = {the male presidents of the United States before 1965} and F = {the female presidents of the United States before 1965}, then $M \cup F = M$. State the property indicated.

7. State which of the following activities are commutative.
   a. Go two blocks west and then three blocks north.
   b. Put on the left shoe and then the right shoe.
   c. Put on socks and then shoes.
   d. Open the door and then walk into the room.
   e. Close the hatch and then submerge the submarine.
   f. Put on the hat and then the jacket.

SET INTERSECTION AND PROPERTIES

In the union, a third set is created from two given sets by pooling together all the elements in each of the two sets. There is another standard way of creating a third set from two given sets. Suppose that one group for reading consists of Charlie, Linus, Lucy, and Snoopy. Suppose also, that one group for mathematics consists of Lucy, Snoopy, Schroeder, Charlotte, and Violet. Then we have two sets,

R = {Charlie, Linus, Lucy, Snoopy}
M = {Lucy, Snoopy, Schroeder, Charlotte, Violet}.

These sets are not disjoint: Lucy and Snoopy are members of both. In fact, common members of two sets suggest a natural set of elements--namely, the set consisting of all members that the sets have in common. Associated with two given sets then, is the set whose members are simultaneously elements of both given sets. This set operation is called the intersection of the two sets, and is denoted by the symbol " $\cap$ ". Thus,

$$R \cap M = \{\text{Lucy, Snoopy}\}.$$

As is the case of the union, the intersection is also uniquely defined. For any two sets, a single set is determined by their intersection.

As $R \cap M$ draws on members of R for its creation, the intersection must necessarily be a subset of R. Likewise, $R \cap M$ must necessarily be a subset of M. Thus, the intersection is a subset of both sets.

Even when two sets are disjoint, we can specify a set of the common elements; this is simply the set that has no members. Recalling that the empty set is a subset of every set, we see that the statement,

THE EMPTY SET IS THE INTERSECTION OF TWO DISJOINT SETS,

is consistent with the observation we have just made, namely, that the intersection must be a subset of each set. Related to this, of course, is that the intersection of a set and its complement is the empty set. This is by virtue of the fact that a set and its complement are disjoint. Another consequence of the fact that the intersection must be a subset of each of its generating sets relates to the intersection of a subset with its super-set. This will be the theme of one of the problems to follow.

PROBLEMS

8. For each of the following problems, state whether C represents the union or the intersection.
   a. $A = \{1, 2, 3, 4\}$, $B = \{2, 4, 5, 9\}$, $C = \{2, 4\}$
   b. $A = \{1, 2, 3\}$, $B = \{4, 5, 6, 7\}$, $C = \{1, 2, 3, 4, 5, 6, 7\}$
   c. $A = \{1, 2, 3\}$, $B = \{4, 5, 6, 7\}$, $C = \{\ \}$
   d. $A = \{1, 2, 3, 4, 5\}$, $B = \{1, 2, 3\}$, $C = \{1, 2, 3\}$
   e. $A = \{1, 2, 3, 4\}$, $B = \{1, 2, 3, 4\}$, $C = \{1, 2, 3, 4\}$
   f. $A = \{1, 2, 3, 4\}$, $B = \{\ \}$, $C = \{\ \}$
   g. $A = \{1, 2, 3, 4\}$, $B = \{\ \}$, $C = \{1, 2, 3, 4\}$
   h. A = {stockholders of Linus Imports, Inc.},
      B = {stockholders of Susan Exports, Ltd.},
      C = {stockholders of both corporations}

i. A = {stockholders of Linus Imports, Inc.},
B = {stockholders of Susan Exports, Ltd.},
C = {stockholders of the Linus-Susan merger}.

9. Find the intersection of each:

a. A = {a, b, h, i, k, o, m, w, z}, B = {c, e, h, j, o, r, w, x},

b. A = {brown-eyed, green-eyed, blue-eyed, raven-haired, brunette, blond, platinum},
B = {pink-eyed, blue-eyed, ox-eyed, black-eyed, red-headed, blond, gray-haired}.

10. If B is a subset of A, what is A ∩ B?

We have examined properties of sets under the union operation in view of possible applications to numbers. Even more, we shall find that these properties are equally applicable to sets of geometric objects. The same is true about properties of sets under the intersection operation. We have seen that if

R = {Charlie, Linus, Lucy, Snoopy},
and M = {Lucy, Snoopy, Schroeder, Charlotte, Violet},
then R ∩ M = {Lucy, Snoopy}.

Note that M ∩ R is also {Lucy, Snoopy}. It is obviously a point in logic that if Lucy and Snoopy are members common to R and M, then these same elements are members that are common to M and R. Our description of this characteristic corresponds to the analogous situation for the union:

THE OPERATION OF INTERSECTION OF SETS IS COMMUTATIVE.

That is to say, under the intersection operation, the order of intersection is immaterial.

Now since the intersection of two sets is a set, we may consider the possibility of intersecting this set with yet another set. To illustrate, suppose F = {a, b, c, d, e, f} and S = {b, d, f, h}. Then

F ∩ S = {a, b, c, d, e, f} ∩ {b, d, f, h}
= {b, d, f}

since b, d, and f are elements of both F and S. If T = {b, c, e, f, h}, then the intersection of F ∩ S with T would be

(F ∩ S) ∩ T = {b, d, f} ∩ {b, c, e, f, h}
= {b, f}.

As before, the parentheses around F and S indicate the grouping of these sets to form the first intersection. Thus, intersection of sets may be formed successively one upon another just as unions may be formed successively. Parallel to our previous investigations of the union, we may pursue the question regarding the result of grouping these same three sets differently. The question then, might be: "How does $F \cap (S \cap T)$ compare with $(F \cap S) \cap T$?" To answer this, first observe that

$$\begin{aligned} S \cap T &= \{b, d, f, h\} \cap \{b, c, e, f, h\} \\ &= \{b, f, h\}. \end{aligned}$$

Therefore,

$$\begin{aligned} F \cap (S \cap T) &= \{a, b, c, d, e, f\} \cap \{b, f, h\} \\ &= \{b, f\}. \end{aligned}$$

This clearly gives the same result that we obtained above for $(F \cap S) \cap T$. In general, we have the associative property under intersection:

FOR SETS A, B, AND C, IT IS TRUE THAT
$(A \cap B) \cap C = A \cap (B \cap C)$.

and we may simplify both of these expressions by dropping the parentheses:

$$(A \cap B) \cap C = A \cap (B \cap C) = A \cap B \cap C.$$

On reflection, this must be so. In all these cases, the final result is the set of all elements that all three sets have in common.

Set intersections will play an important role in our work with numbers and with geometric objects. In particular, when we discuss the rational numbers we will see that they figure very prominently, such as in reducing fractions and finding common denominators.

PROBLEMS

11. If A, B, and C are the sets specified below, illustrate the associative property under intersection by different groupings for $A \cap B \cap C$.

    a. $A = \{1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12\}$,
    $B = \{2, 4, 6, 8, 10\ 12\}$, and $C = \{3, 6, 9, 12\}$.

    b. $A = \{2, 4, 6, 8, 10\ 12\}$ $B = \{3, 6, 9, 12\}$,
    $C = \{2, 4, 7, 11\}$.

    c. $A = \{2, 4, 7, 11\}$, $B = \{3, 6, 9, 12\}$ and
    $C = \{2, 4, 6, 8, 10, 12\}$.

12. If A and B are disjoint, what would be the intersection, $A \cap B \cap C$? What would be the intersection, $A \cap B \cap C \cap D \dots \cap Z$?

THE PRODUCT SET

In the previous sections of this chapter we showed different ways that a third set may be created from two given sets.

There is another way of producing a set from two given sets. This is to form all possible pairs of elements of the two sets. The formation of such sets will be linked directly to multiplication of numbers as well as to graphing.

Suppose in the kindergarten Joe, Mary and Peter can play with blocks, paints, wagon, or turtle. Each child may pick a toy to play with. How many combinations are there?

Joe -- blocks
Joe -- paints
Joe -- wagon
Joe -- turtle

Mary -- blocks
Mary -- paints
Mary -- wagon
Mary -- turtle

Peter -- blocks
Peter -- paints
Peter -- wagon
Peter -- turtle

From this list we see that there are twelve combinations. In this example we have two sets:

C = {Joe, Mary Peter}
T = {blocks, paints, wagon, turtle} .

The combinations of child and toy form a set of all possible pairs in which the first member of the pair is an element of set C and the second member of the pair is an element of T.
If we use initials we have

$$C = \{J, M, P\}$$
$$T = \{b, p, w, t\}.$$

All the combinations (called ordered pairs) we formed are

$(J, b), (J, p), (J, w), (J, t),$
$(M, b), (M, p), (M, w), (M, t),$
$(P, b), (P\ \ p), (P, w), (P, t)$

where (J, b) means the combination Joe-blocks. The set of all these ordered pairs forms a set called the *product set* or *cartesian product*. The product set of C and T is represented by the symbol $C \times T$.

In the product set we have a set whose elements are not single elements but ordered pairs. No member of a product set is a member of either generating set.

Consider

$$A = \{a, b, c\}$$
$$B = \{1, 2\}.$$

Then $A \times B$ is the set

$$\{(a, 1), (a, 2), (b, 1), (b, 2), (c, 1), (c, 2)\}.$$

Now let us form $B \times A$. $B \times A$ is the set

$$\{(1, a), (1, b), (1, c), (2, a), (2, b), (2, c)\}.$$

The pair $(1, a)$ is different from the pair $(a, 1)$. By comparing $A \times B$ and $B \times A$ we see that $A \times B$ is not equal to $B \times A$, but

$$N(A \times B) = N(B \times A).$$

BROADER CONCEPT OF A UNION

In our discussion of the union, the concept of this operation was made on the basis of two disjoint sets. The reason for this restriction is that eventually we intend to link this concept to the addition of whole numbers. Actually, the definition of union does not have this restriction.

> THE UNION OF A AND B IS THE SET
> WHOSE ELEMENTS ARE MEMBERS OF A, OR
> MEMBERS OF B, OR OF BOTH A AND B.

That is to say, elements of $A \cup B$ are members of at least one of the two sets, A, B. With this definition, the concept of a union is broadened to encompass joining sets that have members in common as well as sets that are disjoint. For example, if

A = { ★, ○, △} and B = { ★, △, □, ⌓, ☾},

then

A ∪ B = { ★, ○, △, □, ⌓, ☾}.

Note that the common members ★ and △ are not listed more than once; this in accord with our previous agreement on the specification of a set. The properties that we have noted before under the restricted operation still hold for the broader concept of union.

PROBLEMS

13. For each pair of sets given below, find $A \cup B$, $A \cap B$, and $A \times B$
    a. $A = \{a, b, c, d, e\}$, $B = \{c, e, f\}$
    b. $A = \{c, e, f\}$, $B = \{a, b, c, d, e\}$
    c. $A = \{a, b, c\}$, $B = \{a, b, c\}$
    d. $A = \{a, b, c\}$, $B = \{d, e\}$

14. If A is a set what is $A \cup A$?

SUMMARY OF PROPERTIES

A summary of the properties for sets that we have mentioned in this chapter is catalogued below, where A, B, and C are sets. These are properties that will be particularly meaningful for us when we deal with numbers or sets of points.

1. The union and intersection of sets are commutative :

$$A \cup B = B \cup A \quad \text{and} \quad A \cap B = B \cap A.$$

Note here that the order of operation is immaterial.

2. The union and intersection of sets are associative :

$$(A \cup B) \cup C = A \cup (B \cup C) \quad \text{and}$$
$$(A \cap B) \cap C = A \quad (B \cap C).$$

Note here that the grouping for the operation is immaterial.

3. $A \cup \{\ \} = A.$

APPLICATIONS TO TEACHING

The concept of joining disjoint sets has been reported to be fairly easy for children to comprehend. Apparently, join is a word that is used occasionally in other situations. Sets of buttons, books, or other concrete objects may be joined with other sets of any concrete objects to communicate in a natural way the notion of a union.

The notion of a commutative operation can also be rendered in a concrete form such as books from the shelf joined with books on the desk and books on the desk joined with books on the shelf. In either case, the same set of books is in the union. The words "union" and "commutative" need not be introduced at this point.

The concept of intersection is not introduced until the second grade and formalized in the third grade. However, we can see that it has properties that are analogous to those for the union. We shall make use of the intersection in the next chapter on geometry as well as in our treatment of rational numbers.

The cartesian product will be used here mainly in connection with multiplication and with graphing when we use ordered pairs of numbers to locate points in the plane. The number line will be introduced here along with the presentation of whole numbers in Chapter 7. Eventually, the student will encounter the cartesian product in terms of relations between two sets. The various correspondences between the elements state which elements of one set are related to which elements of another set. Then, a principal undertaking will be to study the characteristics associated with various kinds of relations. From this will evolve the important study of functions. Graphing, of course, gives a pictorial representation of relations. Thus, graphing will be a valuable support for establishing some of the underlying concepts of relations.

As we have hinted in the text, the notion of a relative complement will be applied to our development of the concept of subtraction. In the presentation of these ideas, teachers have reported that they find it helpful to use the words "remove" or "left over" prior to the particular lessons.

QUESTION

"Why is the intersection of two disjoint sets the empty set?"

The intersection of sets consists of all members that the sets have in common. Thus, if $A = \{1, 2, 3, 4\}$ and $B = \{2, 4, 6, 8\}$, then

$$A \cap B = \{2, 4\};$$

2 and 4 are elements of both A and B. Now, if $C = \{1, 2, 3, 4\}$ and $D = \{a, b, c, d\}$, C and D are disjoint because the sets have no members in common. By this token, the set consisting of elements that these two sets have in common is then made up of no members. Another way of stating this is

$$C \cap D = \emptyset.$$

VOCABULARY

Associative
Binary Operation
Cartesian Product*
Commutative
Complement*
Disjoint Sets*
Intersection
Operations
Ordered Pair
Product Set*
Relative Complement of a Set*
Remainder Set*
Remaining Set*
Union*

EXERCISES - CHAPTER 4

1. a. Find the union and the intersection of A and B if $A = \{1, 2, 3, 4, 5\}$ and $B = \{1, 3, 5\}$.
   b. If B is a subset of A, what is $A \cup B$?
   c. If B is a subset of A, what is $A \cap B$?

2. Explain how the union and the intersection of any set with the empty set agree with your findings in Exercise 1.

3. A recipe calls for separating egg whites from egg yolks and emphasizes the particular order these are to be added. Explain the implication of these directions for the cake-mixing operation.

4. State which of the following situations are associative.
   a. Putting peas and carrots together, and adding water.
   b. Eating hot dog, mustard, and coffee.
   c. Paying for groceries with a quarter, a dime, and a nickel.
   d. Putting kerosene with fire, and adding water.

†5. State why the intersection of two sets is always a subset of their union.

SOLUTIONS FOR PROBLEMS

1. a. $\{1, 2\}$
   b. $\{1, 2, 3\}$
   c. $\{1, 2, 3\}$
   d. $\{1, 2, 3, a, b, c, d, e\}$
   e. $\{a, b, c, d, e, 1, 2, 3\} = \{1, 2, 3, a, b, c, d, e\}$; same as in d.
   f. $\{a, b, c, d, e, 1, 2, 3, \alpha, \beta, \gamma, \delta\}$

g. $\{1, 2, 3, \alpha, \beta, \gamma, \delta\}$

h. $\{a, b, c, d, e, 1, 2, 3, \alpha, \beta, \gamma, \delta\}$; same as in f.

2. a. 1c

   b. $A \cup \{\ \} = A$

3. $W \cup X = V$

4. a. Commutative property under union.

   b. Commutative property under union; only order of B and C is changed.

   c. Associative property under union.

5. a. $\{4, 5\}$

   b. $\{4, 5, 9\}$

   c. $\{1, 2, 3, 4, 5, 6, 7, 8\}$

   d. $\{1, 2, 3, 4, 5, 6, 7, 8\}$

6. The union of M and the empty set is M.

7. a. Normally commutative; this depends on the layout of the blocks, also on the location. If the location of the starting point is 3 blocks south of the north pole, then walking 3 blocks north, the north pole is reached. At that point, there is no westerly direction; everywhere is south.

   b. Commutative

   c. Not commutative

   d. Not commutative

   e. Not commutative

   f. Commutative

8. a. Intersection

   b. Union

   c. Intersection

   d. Intersection

   e. Intersection; in a later section of this chapter, "Broader Concept of a Union", it will turn out that this is also the union of A and B.

   f. Intersection

   g. Union

   h. Intersection

   i. Union

9. a. $\{h, o, w\}$

   b. {blue-eyed, blond}

10. If B is a subset of A, then $A \cap B = B$

11. a. $(A \cap B) \cap C = \{2, 4, 6, 8, 10, 12\} \cap \{3, 6, 9, 12\} = \{6, 12\}$
    $A \cap (B \cap C) = \{1, 2, 3, \ldots, 12\} \cap \{6, 12\} = \{6, 12\}$

    b. $(A \cap B) \cap C = \{6, 12\} \cap \{2, 4, 7, 11\} = \{\ \}$
    $A \cap (B \cap C) = \{2, 4, 6, 8, 10, 12\} \cap \{\ \} = \{\ \}$

    c. $(A \cap B) \cap C = \{\ \} \cap \{2, 4, 6, 8, 10, 12\} = \{\ \}$
    $A \cap (B \cap C) = \{2, 4, 7, 11\} \cap \{6, 12\} = \{\ \}$

12. The empty set in either case.

13. a. $A \cup B = (a, b, c, d, e, f)$; $A \cap B = \{c, e\}$;
    $A \times B = \{(a, c), (a, e), (a, f), (b, c), (b, e), (b, f), (c, c), (c, e), (c, f), (d, c), (d, e), (d, f), (e, c), (e, e), (e, f)\}$; the pairs may be listed in a different order; for example, (d, e) may be listed as the first pair. However, the order of the elements within each pair must be observed. (d, e) is a member of $A \times B$, but (e, d) is not a member of $A \times B$.

    b. $A \cup B = \{a, b, c, d, e, f\}$; $A \cap B = \{c, e\}$;
    $A \times B = \{(c, a), (c, b), (c, c), (c, d), (c, e), (e, a), (e, b), (e, c), (e, d), (e, e), (f, a), (f, b), (f, c), (f, d), (f, e)\}$;

    c. $A \cup B = \{a, b, c\}$; $A \cap B = \{a, b, c\}$;
    $A \times B \{(a, a), (a, b), (a, c), (b, a), (b, b), (b, c), (c, a), (c, b), (c, c)\}$;

    d. $A \cup B = \{a, b, c, d, e\}$; $A \cap B = \{\ \}$;
    $A \times B = \{(a, d), (a, e), (b, d), (b, e), (c, d), (c, e)\}$

14. $A \cup A = A$

Chapter 5

INTRODUCTION TO GEOMETRY

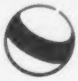
INTRODUCTION

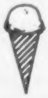
The last chapter dealt with operations on sets to form new sets. The ground now has been prepared for considering sets of points using the set operations to generate various kinds of geometric figures. Besides being a natural direction in which to move at this juncture, geometric figures and certain of their properties will be useful in extending number ideas in the chapters ahead.

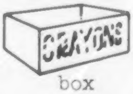
We shall begin our approach to geometry in a way that will be most helpful to teachers of small children. That is, we will consider concrete objects and abstract from them certain desired geometric information. We shall then shift to a more mathematically logical approach.

GEOMETRIC SOLIDS

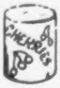
Observation of concrete objects such as those shown below will be helpful for an understanding of abstract representations of figures.


ball


ice cream cone

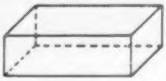


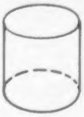
box

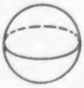


can

Discussion of the characteristics of these shapes facilitates familiarity with some of the vocabulary associated with them.


(a)


(b)

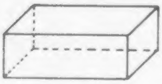

(c)

The above drawings are examples of typical representations of geometric solids. There may be some difficulty in visualizing the 3-dimensional nature of the figures since the drawings are restricted to two dimensions. The dotted lines are included to aid perception. They represent parts of the figures which would not be visible from this vantage point.

In all of these figures, the "inside" is not filled. The object identified by (a) looks like a block. It is not like a block in terms of being composed of matter such as wood. It is shaped like a block but is hollow. Physical objects which can be associated with the geometric solids illustrated are a shoe box, (including the lid), an empty oatmeal box with the lid on, and a balloon. Thus, the word "solid" in "geometric solid" has the mathematical meaning of "three-dimensional" rather than the common usage of "firm" or "not hollow".

The figures (a), (b), and (c) above can be abstracted from numerous physical objects which are available to the teacher and children. Each has characteristics which convey the ideas we want to teach. For example, by looking at and touching models of (a), children will learn to recognize "straight" and "flat" objects with "corners". In (b), they will feel a "rounded" object which also has "edges" and "flat" parts, but no "corners". The third figure illustrates a "rounded" surface without edges or corners.

For our purposes in developing some basic concepts and vocabulary, we will concentrate only on Figure (a). The subject of geometric solids will be developed more thoroughly in a later chapter.

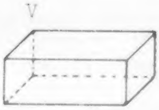


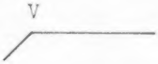
(a)

This "box" (more formally, a rectangular prism) is made of six flat surfaces which are called faces of the prism. The face of a geometric solid is a flat surface of the solid.

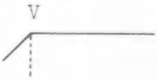
Where two faces meet is an edge of the solid. Each face of this figure has a boundary of four edges. The "skeleton" of the prism is made up of twelve edges.

One other characteristic which we wish to identify in the above solid is that it has "corners" where three edges come together. Each is a vertex (plural: vertices) of the prism. Note that any two of these three edges would meet at the same place and form the same geometric figure. Thus the two figures to the right below





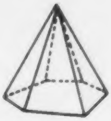
and



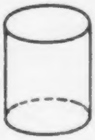
equally well locate the vertex of the prism identified by V. Thus, a vertex may be determined by the meeting of two edges of a face. A point of a geometric figure may sometimes be designated a vertex, however, even though it is not the meeting of two edges of a face. This is the case, for example, with the vertex of a cone.

PROBLEMS*

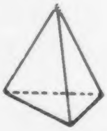
1. How many faces does this solid contain?



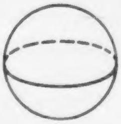
2. Which of the figures have edges but no vertices?



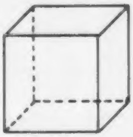
(a)



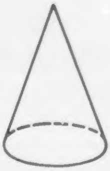
(b)



(c)



(d)



(e)

POINTS AND PATHS

The basic ingredient of all geometric configurations is what is called a point. A point may be thought of as a precise location. Points are represented by dots on a paper or as the end of a sharply pointed pencil. All of these are visual aids to assist us in conceptualizing the nature of a point.

These representations are merely attempts to symbolize the idealized geometric entity called a point. The difficulty is that a point is an idea rather than a physical object. The point which we represent by a dot, no matter how small the dot, covers many locations.

When we arrive at the description of a point as an exact location, this is not a definition of a point in the formal sense. If we say a point is an exact location, "exact location" must be understood. The dictionary might define location as a "position in space". Position in space might refer us back to point. If none of these words were meaningful to us, the dictionary would hardly clarify matters. However, the circularity in dictionary definitions is necessary because there is only a finite number of words accessible in the dictionary. Eventually some word in the chain of definitions must reappear. Implicit in this is that at least one word in the chain must be simply understood so that others may be defined in terms of it. "Point" is such a word in geometry. In a sense it is the "first" word in the vocabulary of geometry, and we say it is an undefined term.

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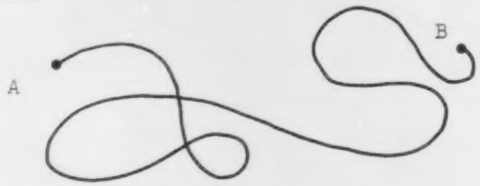
*Solutions to problems in this chapter are on page 84.

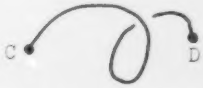
Once the concept of point is understood, we will again rely on representing points by marks on paper to facilitate discussing them. They are commonly labeled by capital letters. The drawing represents point P, or simply P, by which a point is understood.

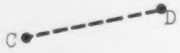
Every geometric figure is a set of points. A curve is a set of points followed in moving along a path from one point to another.



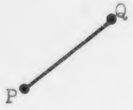
Thus the drawing above represents a path from point A to point B, or from point B to point A. It is evident that there are other curves from A to B; indeed there are infinitely many.

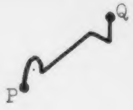
Inherent in the notion of path is the idea of continuity. There may not be gaps in a path. Neither of the drawings below is a path from C to D.

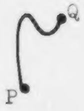




According to the strict mathematical definition, curves do not have to be continuous. We, however, will consider only those that are. Hereafter, by "curve" we shall mean a continuous curve. Portions of the path or the entire path may be straight. As a path may be used to specify the set of points in a curve, any of the following figures represents a curve from P to Q.

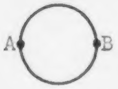




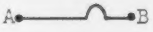


PROBLEM

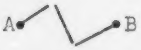
3. State whether or not each of the following figures represents a curve from A to B.



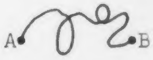
(a)



(b)



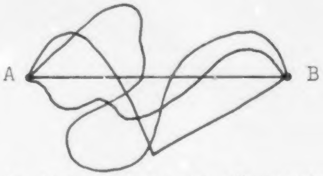
(c)



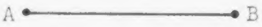
(d)

LINE SEGMENTS

Let us represent two points by the dots below labeled A and B. We now trace several paths from point A to point B as shown above.



One of the paths shown in the picture is of special importance. It is the most direct path from A to B. This path, represented below, is called a line segment.



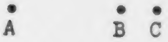
The symbol for this line segment is $\overline{AB}$ or $\overline{BA}$ and the points A and B are called the endpoints of $\overline{AB}$. A line segment is named by its two endpoints. Since both $\overline{AB}$ and $\overline{BA}$ denote the same segment, the order in which the endpoints are named is irrelevant. With the concept of line segment, we can now identify an edge of a rectangular prism as a line segment.

PROBLEMS

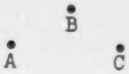
4. Represent $\overline{AB}$ such that it can also be named as the union of $\overline{AQ}$, $\overline{QM}$, and $\overline{MB}$.

5. In the same $\overline{AB}$ of problem 4, assume Q is between A and M. State all other possible relationships of one point being between two other points.

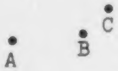
By $\overline{AB}$ is implied the set of points, A, B, and all points between them. Thus the notion of betweenness is intuitively derived. However, there may be need to clarify what is meant by "between". If A, B, C, are three points as indicated in (a), it may be quite natural to consider that B is between A and C. Even if the points were as in (b) or (c) one might concede that B is between A and C. But if the three points were as in (d), the question as to which point is between which other two points is not so easily resolved.



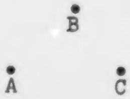
(a)



(b)

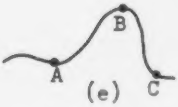


(c)

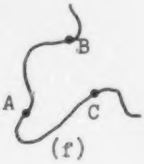


(d)

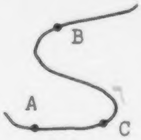
Implicit in the decision as to which point is between two others apparently is a curve connecting these points. If a curve passes through the three poins as in (e), then B is between A and C.



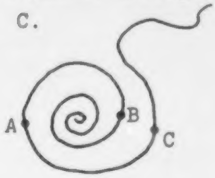
(e)



(f)



(g)



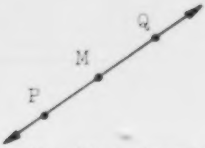
(h)

In (f), A is between B and C, and in (g), C is between A and B. This, of course, can be done for points which we may have considered sufficiently clear with regards to betweenness. Thus, in (h), A is between B and C. Eventually, a number will be associated with each pair of points. We will call this number the distance from one point to the other. Betweenness can then be stated in terms of distances. Even with this definition, a curve is involved in the concept of distance. The common sense interpretation of betweenness, when no curve is specified, is simply that the points are to lie along a straight path. When we say that a point is between two others, it will be our understanding then, that the three points are all on the same line segment.

LINE

Once a line segment is defined by the location of its two endpoints and all the points between them, it determines two directions. If we imagine extending a given segment infinitely far in both of these directions, we conceive of a geometric *line*.

A circularity will be noted in defining a line and betweenness. A *line* is conceived of as an extension of a segment, and a *segment* is defined as the set consisting of two endpoints and all the points between. On the other hand, *between* is stated in terms of points on a line. This circularity is unavoidable in definitions, and ultimately, we must accept these notions as primitive and undefined. Thus, in geometry, a line is accepted simply as: "a certain set of points".



The drawing represents the line formed by extending $\overline{PQ}$ in both of its determined directions. The arrowheads are used to indicate that the extension is infinite. We adopt the notation $\overleftrightarrow{PQ}$ for the line containing the two points P and Q in order to distinguish it from line segment PQ written as $\overline{PQ}$. We could also refer to the line as $\overleftrightarrow{PM}$, $\overleftrightarrow{MQ}$, $\overleftrightarrow{QM}$, and so on. In general, any "two points in the set of points" in the line may be used to name it. Again, order does not matter.

It is important not to use this terminology loosely. A line has no endpoints, while a line segment must have two endpoints.

SPACE

Now that we understand the geometric concept of a point, we may now define *geometric space* or simply *space* as the set of all points.

The usual connotation of space is the set of all points in a three-dimension extent. The notion of space in the more general sense, as, simply, a set of all points, is extended to the branches of mathematics other than geometry. Thus, in probability, the set of all possible outcomes of a certain definition is described as the sample space. The meaning of space is generally determined by the context in which it is used. Unless otherwise indicated, space in this text will refer to infinite, three-dimensional space.

PLANE

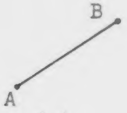
Let us now consider a subset of the set of points of space called a plane. Again we do not give a formal definition of the plane.

Any flat surface such as the floor, the top of the desk or a piece of paper suggests the idea of a plane. Like the line, a plane is unlimited in extent. That is, any flat surface used to represent a plane only represents a portion of the plane.

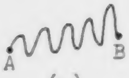
In teaching children, to express the meaning of plane, such objects as the floor, tabletops, and faces of blocks should be examined and felt. The notion of the infinite extent of the plane is approached by thinking in terms of an ever-expanding tabletop and so on.

SIMPLE CLOSED CURVES

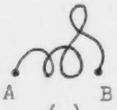
In our discussion of segments, we considered paths between two points and observed that each of the paths describes a curve. A path thus specifies a set of points known as a curve from A to B. When A and B coincide, the curve is said to be closed. Thus, each of the diagrams illustrated represents a curve. The ones appearing on the second row are closed curves.



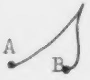
(a)



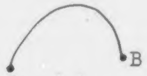
(b)



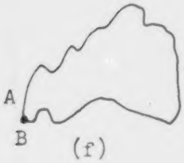
(c)



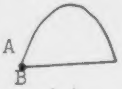
(d)



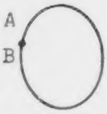
(e)



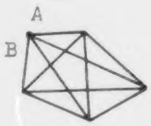
(f)



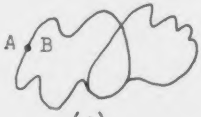
(g)



(h)



(i)



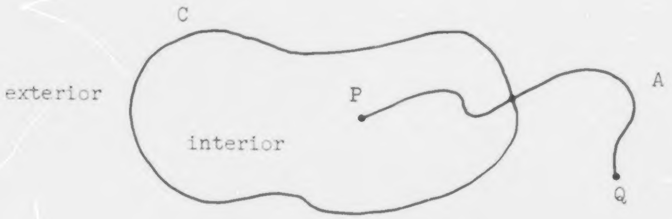
(j)

Of the closed curves that we have drawn, the first three are distinguished from the last two. None of the first three curves crosses itself. To describe the fact that the curve does not cross itself, we say it is simple. By simple closed curve we shall mean a set of points in a plane represented by a path that begins and ends at the same point and does not cross itself.

Simple closed curves have the important property of separating the rest of the plane into two disjoint subsets, the interior (the subset of the plane enclosed by the curve) and the exterior. Thus, with a simple closed curve there is a natural partitioning of a plane into three disjoint subsets:

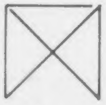
(1) the set of points that are enclosed by the curve
(2) the set of points that are on the curve
(3) the set of points that are neither enclosed by the curve nor on the curve.

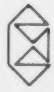
With the separation, any curve in the plane connecting a point of the interior with a point of the exterior necessarily intersects the simple closed curve. This is illustrated by the figure below, where C is the simple closed curve, P is an interior point, Q is an exterior point, and A is a plane curve connecting P and Q.

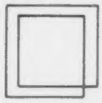


PROBLEMS

6. Which of the following curves are simple?


(a)


(b)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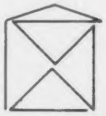

(d)

7. Which of the above curves are closed? Which is simple and closed?

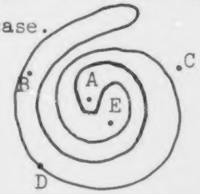
8. Which of the curves in Problem 6 is a union of two simple closed curves?

9. If the letters of the alphabet were printed in block type without serifs (no "tails"), which letters indicate simple closed curves?

ABCD...

Which are simple but not closed?

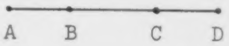
10. a. Can a curve be drawn from A to B without crossing the given curve? from A to C? from A to D? from A to E? State the reason for each case.



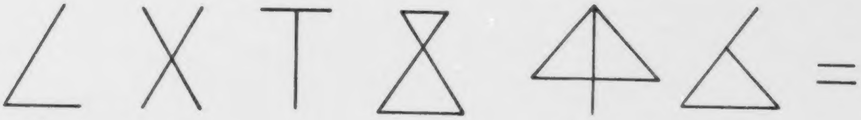
b. From which point in the above diagram is it impossible to draw a curve to any other point without intersecting the curve?

POLYGONS

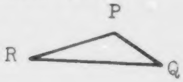
An important class of simple closed curves is the class of polygons. A polygon is a simple closed curve that is a union of line segments. Not all unions of segments form simple closed curves. For example, the union of two segments may again be a segment. In the picture below, the union



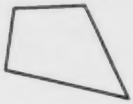
of $\overline{AC}$ and $\overline{BD}$ is $\overline{AD}$; the union is simple, but not closed. Nor is any of the figures below a simple closed curve although each is a union of line segments. Triangles, quadrilaterals, pentagons, and so on,



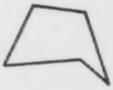
are examples of polygons. Note that $\overline{AD}$ above contains many other segments. For example, $\overline{AB}$ is contained in $\overline{AD}$, $\overline{BC}$ is contained in $\overline{AD}$, $\overline{AD}$ is contained in itself, and so on. Likewise, with segments of a polygon, segments are contained in segments. If a segment of a polygon is contained in no segment other than itself, then this segment is called a side of the polygon. For example, $\overline{PR}$ is a side of the triangle shown below.



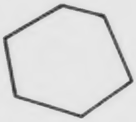
triangle



quadrilateral



pentagon



hexagon

A polygon of three sides is a triangle; four sides, a quadrilateral; five-sides, a pentagon; six sides, a hexagon; and so on. The endpoints of the sides are the vertices of the polygon. Note that each vertex is a common endpoint of two sides. Note also that the number of sides is the same as the number of vertices.

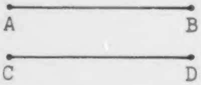
CONGRUENT SEGMENTS

Congruence is a very important and complex idea with many consequences in geometry. We shall confine ourselves to an intuitive approach to the idea of congruence. That is, if one geometric configuration is an exact copy of another, we shall say that the two figures are congruent.

To decide whether two segments are congruent, we can make a tracing of one and see whether or not the tracing fits exactly on the other. If they fit exactly, the segments are said to be congruent. It is, in this sense, that markings on a ruler perform the function of the movable copies of segments.

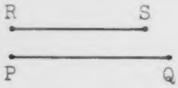
PROBLEMS

11. Make a tracing of $\overline{CD}$. Fit this copy on $\overline{AB}$ to see whether or not $\overline{AB}$ and $\overline{CD}$ are congruent.

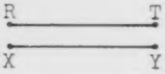


12. Which of the following pairs of line segments are congruent?

a.

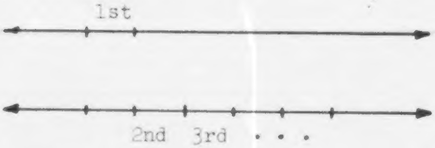


b.

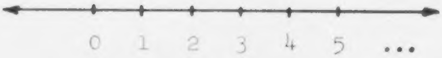


THE NUMBER LINE

Congruent segments give us a way of relating numbers with points on a line. This is the case with the number line. Given any two points on a line, a segment is determined. We can continue to mark off points, one after another so that each segment is congruent to the first.



The points may be labelled 0, 1, 2, 3, 4, ..., in the order of the whole numbers. Although one can assign these labels from right to left, conventionally we proceed from left to right. When points are labelled thus, the numbers associated with the points are called the coordinates of the points, and the line together with its coordinates is called the number line.



The Number Line

The number line thus gives us a 1-1 correspondence between the set of endpoints of congruent segments and the set of whole numbers. That is, each endpoint is associated with one and only one whole number, and each whole number is associated with one and only one endpoint of the congruent segments on the line. This device is quite useful for us. It enables us to visualize the order of numbers by the position of corresponding points on the line. We will later connect operations in arithmetic with operations on the number line.

PROBLEMS

13. What is the smallest whole number represented on the number line?

14. What can you say about every number represented by a point on the number line that lies to the right of a given point?

APPLICATION TO TEACHING

Logically, as geometric figures are made up of points, one should begin the study of geometry with the concept of what constitutes a point. Lines, curves, planes, solids, and spaces may be generated from a point.

Despite the logical basis, the sets of geometric objects that children have to manipulate are sets of three-dimensional objects. These are the concrete objects which provide children with experiences from which they can abstract the mathematical concepts. For this reason, we begin with models of solids. From the models, we identify faces, edges, and vertices. Once identified, we can use these primitive elements to construct other geometric figures. For example, "skeletons" of pyramids and prisms are unions of certain line segments.

For children, the approach to closed figures is entirely geometric. It must be emphasized that any closed figure that does not lie in a plane is called a "solid", even though it is hollow. For example, a rectangular "box" consisting only of the faces is a "solid"; the "skeleton" of a rectangular box is a "solid".

It is a good idea to display a set of wooden models that are not too small and encourage the children to examine and handle them for several days before beginning the chapter, "Recognizing Geometric Figures." Tracings of the faces of the solids may be made on a large sheet of paper and displayed so that the children may match a face of a solid to its tracing. Wire or stick models of polygons whose sides are congruent to edges of solids may be used for the same purpose. Matching pictures of the solids with the appropriate models should prove useful in helping the children to visualize drawings of 3-dimensional solids. Most pupils seem to be interested in finding objects at home which qualify as cylinders and rectangular boxes and so on.

Solid figures may be identified as blocks, boxes, or balls. For example, a triangular pyramid may be referred to as a block with triangular faces; but it would not be appropriate to identify a ball as a circle or a rectangular prism as a rectangle. Basic distinctions to be made for the children at

straight edge vs. rounded edge;
flat region vs. rounded region;
flat figure vs. solid figure.

We have stated that in the study of geometry, each of the following objects, a point, a line, and a plane, may be regarded as a primitive element. By these, we can define other geometric objects. Likewise, a 3-space may serve as a primitive element, and it is from this standpoint that we consider points, lines, planes, spaces, as elements of geometry.

QUESTION

"What is meant by the statement that space is the set of all points?"

Using various physical objects as models, we might move to an ever-shrinking dot as representation of a point. The concept of point that we have in mind is thus an abstraction from visual models to an idealized concept: that of an exact position having no length, no width. In other words, having no dimensions. The space that we commonly think of as three-dimensional space can be visualized as the set of all possible positions in this space. Certainly we can imagine that wherever there is a location, we can visualize a point; the totality of all positions thus fills up the space all about us. This is an account for our definition of space as the set of all points. However, the definition is a good deal more far-reaching than this. The word "all" may have many frames of reference.

If we think of the plane as the idealizing of an ever-expanding table, we "see" that this too is composed of a collection of points; namely, all the points located on the imagined surface. With reference to this surface, the collection of all points so restricted is also a space. This is the space we call the plane. Because the plane occupies an extent along a dimension which we might conceive of as length and along one which we might conceive of as width, the plane is said to extend along two dimensions. Another way of saying this is : the plane is two-dimensional space (one that has no thickness). Similarly space in a line is one-dimensional. So, space is dependent upon how much this "all" encompasses; whatever the case, it is the set of all points. In usual terminology and without other qualification, by "space", the understanding is the space of three dimensions is meant.

VOCABULARY

Between*
Closed Curve*
Congruence*
Congruent
Congruent Segments*
Coordinates*
Curve*
Edge*
Endpoints
Exterior (outside) of a Simple Closed Curve*
Face
Geometric Solids
Hexagon*
Interior (inside) of a Simple Closed Curve*
Line *
Line Segment*
Number Line*
Path*
Pentagon*
Plane*
Point
Polygon*
Quadrilateral*
Rectangular Prism
Simple Closed Curve*
Space
Triangle*
Vertex of a Polygon*
Vertex of a Prism*

EXERCISES - CHAPTER 5

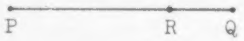
1. Draw a representation of a geometric solid shaped like the pyramids of Egypt. How many vertices does it have?

2. Explain the differences between $\overline{AB}$ and $\overleftrightarrow{AB}$.

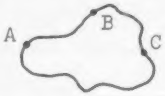
3. How many different lines may contain:
   a. one certain point?
   b. one certain pair of points?

4. If any of the following statements are false, rewrite them correctly.
   a. Two points determine a line segment.
   b. Three points determine a plane.
   c. The intersection of two planes may be a line.

5. Show why $\overline{PQ}$ cannot be divided into disjoint segments so that the union is $\overline{PQ}$.



6. a. In the following figure, which of the points, A, B, C, is between the other two?



   b. If three points are connected by a curve, is one point necessarily between the other two?

7. a. If a railroad does not have spur tracks and does not cross itself, what points form the boundary restricting the extent of a train's journey?
   b. To restrict the extent of a ship's operation, what kind of boundary might be required?
   c. What kind of boundary might be required to restrict the extent of an airplane operation?
   d. What kind of boundary might be required to restrict the extent of a submarine operation?

8. If one number is greater than another; what do you know about their positions on the number line?

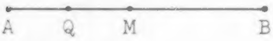
9. a. According to the outlines for Books K-3 in Appendix A, in which grades are closed curves presented formally as a topic?
   b. In which grades are topics discussed using basic concepts of closed curves?

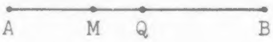
SOLUTIONS FOR PROBLEMS

1. 7

2. a. only. The rounded surface intersects the two faces in two edges; b. and d. have both vertices and faces; c. has neither; e. has a vertex as well as a face.

3. (b) and (d) represent curves from A to B; (a) represents a curve from A to A or from B to B. There are two curves represented from A to B, however; (c) is not a curve from A to B, it is not continuous.

4. 

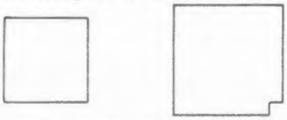
or


5. Figure corresponds to the first possibility shown in 4. Q is between A and B: M is between A and B; M is between Q and B.

6. (b) and (d); The curves (a) and (c) cross themselves once and so are not simple.

7. (b) and (c); (b) is the only curve both simple and closed.

8. (c) The two simple closed curves whose union is figure (c) are



9. D, O are simple closed curves.

   C, G, I, J, L, M, N, S, U, V, W, Z are simple but not closed.

10. a. The curve is simple and closed; therefore the plane is separated into 3 disjoint sets; the interior of the curve, the curve, and the exterior of the curve. E and B are in the interior; D is in the curve; A and C are in the exterior.

    Thus no curve can connect A to B, A to D, or A to E without crossing the curve. A and C can be connected by such a curve, however.

    b. D. any curve that contains D intersects the curve at least once, namely at D.

11. $\overline{AB}$ is congruent to $\overline{CD}$.

12. (b)

13. 0

14. The coordinate of every point to the right of a given point is greater than the coordinate of the given point.

Chapter 6

NUMERATION: NAMING NUMBERS

INTRODUCTION

In this chapter we shall consider explicitly the important distinction between numbers and their names. We shall concentrate our attention to schemes for naming whole numbers; that is, to the problem of numeration.

WHOLE NUMBERS AND THEIR NAMES

We know that the whole number "twelve", for example, is a property of the set

$$\{a, b, c, d, e, f, g, h, i, j, k, l\}$$

and of all sets equivalent to this set. The word "twelve" is a name for this number property and is not the number itself. Similarly, the symbol or numeral "12" is another name for this same number. This is true also for the numeral "XII", written in the Roman system of notation. In fact, when we write

$$\mathrm{XII} = 12$$

we simply are asserting that "XII" and "12" are two different names for the same thing; that is, names for the same number.

As we now consider principles of numeration, it is important for us to keep clearly in mind that number and numeral are not synonymous. A number is a concept, an abstraction. A whole number is one kind of number, and in various preceding chapters we have considered selected aspects of the whole number system. On the other hand, a numeration system is a system for naming numbers; thus, it is a numeral system. In this chapter, we shall be concerned with numeration systems for naming whole numbers. Our emphasis will be on the number names or numerals, rather than on the numbers themselves.

ANCIENT NUMERATION SYSTEMS

Man, during the course of his history, did not always use our familiar Hindu-Arabic numeration system. His earliest schemes involved little more than tally marks, such as / for "one", // for "two",

/// for "three", etc. Such primitive schemes were far from effective and efficient, particularly when dealing with large numbers.

The Egyptians, the Chinese, the Greeks, the Romans, and others all developed numeration systems that were improvements upon primitive tally schemes. However, none of these was as sophisticated as the one developed by the Hindus, which evolved into the Hindu-Arabic system we use today. Nevertheless, a brief consideration of at least one of these earlier numeration systems can be of interest and can give us an appreciation of the principles and advantages of our own system.

A MODIFIED GREEK SYSTEM

One of the Greek systems of numeration used twenty-seven basic symbols: the twenty-four letters of the Greek alphabet, an obsolete letter, and two letters borrowed from the Phoenicians. Each of these basic symbols named a particular number. Other numbers were named by combining basic symbols according to established principles or "rules".

Let us illustrate a modified version of this Greek system by using as basic symbols the twenty-six letters of our own alphabet and one additional arbitrary symbol, ▽ . The number named by each basic symbol is indicated below in terms of our own Hindu-Arabic numerals.

A = 1
B = 2
C = 3
D = 4
E = 5
F = 6
G = 7
H = 8
I = 9

J = 10
K = 20
L = 30
M = 40
N = 50
O = 60
P = 70
Q = 80
R = 90

S = 100
T = 200
U = 300
V = 400
W = 500
X = 600
Y = 700
Z = 800
▽ = 900

A compound symbol such as "PD" is interpreted to mean

70 + 4, or 74

in our own system. Similarly,

"WKH" means 500 + 20 + 8, or 528,
"TR" means 200 + 90, or 290, and
"UF" means 300 + 6, or 306

in terms of our familiar numerals.

Notice that the symbol "DP" would be interpreted to mean 4 + 70, or 74. Thus, it would be true that

PD = DP.

However, we shall agree that in such instances we shall write the basic symbol for the larger number to the left of the basic symbol for the smaller number. Thus, the preferred form would be PD instead of DP. Similarly, it would be true that

WKH = WHK = KHW = HWK = HKW = KWH.

Of these six different names for the same number, the preferred form would be WKH.

PROBLEMS *

1. Express each of these modified Greek system numerals as familiar Hindu-Arabic numerals.

   a. MG b. ZNB c. XK d. VC e. ▽RI

2. Express each of these Hindu-Arabic numerals in the "preferred form" of modified Greek system numerals.

   a. 63 b. 735 c. 210 d. 504 e. 888

3. Does the modified Greek system have a basic symbol for the number "zero"? If so, what is that symbol? If not, why is such a basic symbol not used in the system?

But what about naming numbers greater than ▽RI, or 999? We cannot name such numbers without some further agreement or extension

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* Solutions for problems in this chapter are on page 110.

of the system. So, let us agree that we may use a slash mark (/) to indicate that the number named by a basic symbol is to be multiplied by one thousand (1000). Thus,

/ E means 1000 × 5, or 5000,
/ P means 1000 × 70, or 70,000,
and / T means 1000 × 200, or 200,000

in terms of our familiar numerals. In a numeral composed of a collection of symbols, the slash mark refers to only the one symbol that is immediately to its right.

PROBLEMS

4. Express each of these modified Greek system numerals as familiar Hindu-Arabic numerals.

a. /BYMG   b. /Q/AUL   c. /V/ORC

You undoubtedly have noticed that the number "ten" is of particular significance in the modified Greek numeration system. For instance, the symbols J, K, ..., Q, R named multiples of ten (10, 20, ..., 80, 90), and the symbols S, T, ..., Z, ▽ named multiples of ten tens or one hundred (100, 200, ..., 800, 900).

We may say that "ten" is the base of this numeration system. It is the basic number that we use for groupings within the system.

FEATURES OF NUMERATION SYSTEMS

Many numeration systems have three features that are of significance as we turn to a consideration of our own Hindu-Arabic system.

1. One of these features is that of base, a basic number in terms of which we effect groupings within the system. This number may or may not be "ten". If the base is "ten", we often refer to that system as a decimal system. ("Decimal" is derived from the Latin word decem which means "ten".)

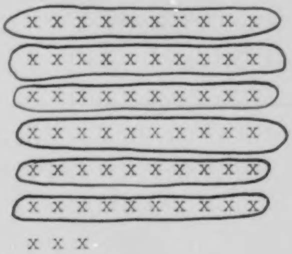
2. Another feature is a set of basic symbols or number names. From these, all other numerals are built. As we shall see, the choice of base often determines the number of basic symbols used within a numeration system.

3. A third feature is a set of principles or rules for combining basic symbols to form other numerals so that every whole number may be named in terms of these basic symbols only. It is within this third feature that we find a principle that sets the Hindu-Arabic system apart from others that preceded it. We are referring, of course, to the principle of place value.

THE HINDU-ARABIC NUMERATION SYSTEM

Let us examine each of the preceding features as it relates specifically to our Hindu-Arabic numeration system.

1. The Hindu-Arabic numeration system is a decimal system: its base is ten. This is seen clearly in the fact that we interpret the number "sixty-three", for example, as "six tens and three (ones)". "Sixty" itself means "six tens". This feature may be illustrated in the groupings below for the interpretation of the number "sixty-three".

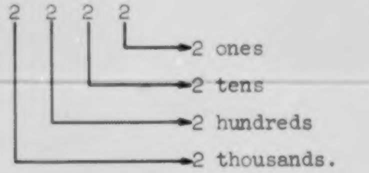


2. The Hindu-Arabic numeration system utilizes ten basic symbols or digits: 0, 1, 2, 3, 4, 5, 6, 7, 8, 9 such that

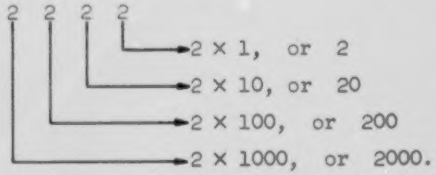
0 names the number zero;
1 names the number one;
2 names the number two;
3 names the number three;
4 names the number four;
5 names the number five;
6 names the number six;
7 names the numver seven;
8 names the number eight;
and 9 names the number nine.

Notice the inclusion of a symbol for zero: 0. This is in marked contrast to systems such as the Greek, the Roman, etc., that had no zero symbol. The need for a zero symbol in the case of the Hindu-Arabic system is related closely to the place value principle discussed in the following section.

3. The Hindu-Arabic numeration system utilizes a principle of place value, along with principles of addition and multiplication, in order to combine basic symbols or digits of the system to name whole numbers greater than nine. We are quite familiar with the fact that in the numeral 2222, for instance, each digit 2 does not have the same "value". The "value" of each 2 is determined by its place or position in the numeral as a whole:



Or, we may convey the same idea in a slightly different way:

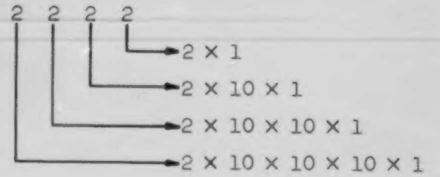


Here we see the principle of multiplication in association with the place-value principle.

We frequently find it helpful to use an expanded form of notation to emphasize both the multiplicative and additive principles that apply to the interpretation of a numeral such as 2222:

$$2222 = (2 \times 1000) + (2 \times 100) + (2 \times 10) + (2 \times 1).$$

None of the notations used thus far has made explicit the important role of the base, ten, in determining the "place values". Each place to the left of the ones place in a numeral has associated with it a "value" that is ten times the "value" associated with the place immediately to its right. For the numeral 2222, we can show this important idea in this way:



or

$$2222 = (2 \times 10 \times 10 \times 10) + (2 \times 10 \times 10) + (2 \times 10) + (2 \times 1).$$

The importance of the zero symbol, 0, In connection with our place-value numeration system is reflected in numerals such as 2220, 2202, 2022, 2200, and 2002. Without the zero symbol such numerals could not be distinguished readily from 222 (in the case of 2220, 2202, and 2022) or from 22 (in the case of 2200, 2020, and 2002). Without some symbol to denote "not any" in a particular place, a numeration system with a place value principle would not be feasible. In fact, the relatively late invention of a symbol for "not any" (a symbol for the number pertaining to the empty set), was the reason for the relatively late creation of a place-value numeration system.

The following chart may be helpful in summarizing some of the ideas just discussed regarding our numeration system.

Millions		Thousands			Units		
tens	ones	hundreds	tens	ones	hundreds	tens	ones
10,000,000	1,000,000	100,000	10,000	1,000	100	10	1
$10\times\underline{10\times10\times10\times10\times10\times10}$	$10\times\underline{10\times10\times10\times10\times10}$	$10\times\underline{10\times10\times10\times10}$	$10\times\underline{10\times10\times10}$	$10\times\underline{10\times10}.$	$10\times\underline{10}$	$10\times\underline{1}$	1
	7	2	0	5	0	4	6

Consider the numeral 7, 205,046 which we read as: "seven million, two hundred five thousand, forty-six". (Notice that the word "and" is not used in reading numerals for whole numbers. Otherwise, it would not be clear, for example, when we say "two hundred and five thousand" whether we mean "200 + 5,000" or "205,000".)

We may interpret the numeral 7,205,046 to mean: 7 millions, 2 hundred-thousands, 0 ten-thousands, 5 thousands, 0 hundreds, 4 tens, 6 ones. Since 0 ten-thousands and 0 hundreds both result in zero, these may be omitted in the interpretation. Thus, 7,205,046 means: 7 millions, 2 hundred-thousands, 5 thousands, 4 tens, 6 ones. We also may use an expaned notation form:

7,000,000 + 200,000 + 5,000 + 40 + 6, or

(7 × 1,000,000) + (2 × 100,000) + (5 × 1,000) + (4 × 10) + (6 × 1), or

(7 × 10 × 10 × 10 × 10 × 10 × 10) + (2 × 10 × 10 × 10 × 10 × 10) + (5 × 10 × 10 × 10) + (4 × 10) + (6 × 1).

PROBLEMS

5. Write the base ten numeral for each of these expressions.

   a. 7 hundreds, 4 tens, 9 ones.

   b. 8 thousands, 3 hundreds, 6 ones.

   c. $2000 + 700 + 50 + 1$

   d. $40{,}000 + 6000 + 80 + 3$

   e. $(5 \times 1000) + (0 \times 100) + (2 \times 10) + (4 \times 1)$

   f. $(7 \times 10{,}000) + (6 \times 100) + (9 \times 1)$

   g. $(8 \times 10 \times 10 \times 10) + (4 \times 10 \times 10) + (3 \times 1)$

   h. $(9 \times 10 \times 10 \times 10 \times 10) + (5 \times 10 \times 10 \times 10) + (6 \times 10)$

6. Express each of these base ten numerals in three ways as shown in the illustrative example below.

   Example: $4257 = 4000 + 200 + 50 + 7$

   $4257 = (4 \times 1000) + (2 \times 100) + (5 \times 10) + (7 \times 1)$

   $4257 = (4 \times 10 \times 10 \times 10) + (2 \times 10 \times 10) + (5 \times 10) + (7 \times 1)$

   a. 6184 b. 7350 c. 40,702

GROUPING BY FOURS (BASE-FOUR NUMERATION SYSTEM)

We are familiar with grouping objects by tens in connection with our decimal place value numeration system. For instance:

	Number of		Base Ten Numeral
	Tens	Ones	
x		1	1
xx		2	2
xxx		3	3
xxxx		4	4
xxxxx		5	5
xxxxxx		6	6
xxxxxxx		7	7
xxxxxxxx		8	8
xxxxxxxxx		9	9
$\boxed{\text{xxxxxxxxxx}}$	1	0	10
$\boxed{\text{xxxxxxxxxx}}$ x	1	1	11
$\boxed{\text{xxxxxxxxxx}}$ xx	1	2	12
$\boxed{\text{xxxxxxxxxx}}$ xxx	1	3	13
$\boxed{\text{xxxxxxxxxx}}$ xxxx	1	4	14
$\boxed{\text{xxxxxxxxxx}}$ xxxxx	1	5	15
$\boxed{\text{xxxxxxxxxx}}$ xxx $\boxed{\text{xxxxxxxxxx}}$	2	3	23

Etc.

Suppose that we agreed to group objects by fours rather than by tens. Suppose, for example, that instead of grouping fourteen objects as

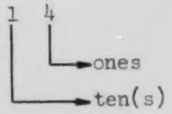
$\boxed{\text{xxxxxxxxxx}}$ xxxx   1 ten and 4 ones

we had grouped the fourteen objects as

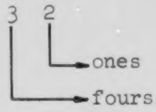
$\boxed{\text{xxxx}}$ $\boxed{\text{xxxx}}$ $\boxed{\text{xxxx}}$ xx   3 fours and 2 ones.

We certainly have *not* changed the *number* of objects: fourteen. We have only changed the way in which these fourteen objects are grouped: as "3 fours and 2 ones" rather than as "1 ten and 4 ones".

The numerals of our base ten place value system reflect a tens-and-ones grouping, as



Would it be possible to develop a base-four place-value numeration scheme whose numerals reflect a fours-and-ones grouping, as



Let us use sets of one, two, three, ..., fifteen objects to see how such a base four numeral system might be developed. This is done in the chart on the opposite page which includes contrasting base ten numerals.

Note that in the decimal system, each set of ten objects is grouped as 1 ten and the number of these groups is indicated in the tens place. Thus, 23 is 2 tens and 3 ones, and the number of ones left ungrouped is given by the digit 3. The possible digits in the ones place are then any of the numerals 0, 1, 2, 3, ..., 9. Similarly, groups of tens are regrouped into hundreds when there are ten or more of these groups, groups of hundreds are regrouped into thousands when there are ten or more of the hundreds, and so on. Thus, any digit in any place is one of the numerals 0, 1, 2, 3, ..., 9. A similar analysis shows that any digit in base four numeration system is one of the numbers 0, 1, 2, 3 since any number of groups exceeding 3 would be regrouped into groups of the next larger size.

	Number of		Base Four Numeral	Base Ten Numeral
	Fours	Ones		
x		1	1	1
xx		2	2	2
xxx		3	3	3
$\boxed{xxxx}$	1	0	10*	4
$\boxed{xxxx}$ x	1	1	11	5
$\boxed{xxxx}$ xx	1	2	12	6
$\boxed{xxxx}$ xxx	1	3	13	7
$\boxed{xxxx}$ $\boxed{xxxx}$	2	0	20	8
$\boxed{xxxx}$ $\boxed{xxxx}$ x	2	1	21	9
$\boxed{xxxx}$ $\boxed{xxxx}$ xx	2	2	22	10
$\boxed{xxxx}$ $\boxed{xxxx}$ xxx	2	3	23	11
$\boxed{xxxx}$ $\boxed{xxxx}$ $\boxed{xxxx}$	3	0	30	12
$\boxed{xxxx}$ $\boxed{xxxx}$ $\boxed{xxxx}$ x	3	1	31	13
$\boxed{xxxx}$ $\boxed{xxxx}$ $\boxed{xxxx}$ xx	3	2	32	14
$\boxed{xxxx}$ $\boxed{xxxx}$ $\boxed{xxxx}$ xxx	3	3	33	15

We now face a problem. What, for instance, does the numeral "13" mean: "1 ten and 3 ones" or "1 four and 3 ones"? We commonly resolve this problem in the following way.

If we see the numeral "13", for example, we assume that it is written in base ten and understand it to mean "1 ten and 3 ones". This simply follows familiar convention.

If, on the other hand, we wish to write a numeral to convey a base four grouping such as "1 *four* and 3 ones" we agree to use the form "13_{four}". The subscript "four" indicates the base in which the numeral is written.

---

*This numeral should be read: *one*, *zero*, *base* *four*. Succeeding numerals in this column would be read: *one*, *one*, *base* *four*; *one*, *two*, *base* *four*; *one*, *three*, *base* *four*; etc.

On occasion, when showing the base ten numeral for thirteen, for instance, we may write "13_{ten}" instead of simply "13", just to be certain that there is no misunderstanding. Thus, we agree that

$$13 = 13_{ten}.$$

However, be sure to keep clearly in mind that

$$13 \neq 13_{four}^{*}$$

and that

$$13_{ten} \neq 13_{four}.$$

In fact, it is true that

$$13_{ten} = 31_{four}.$$

and that

$$13_{four} = 7_{ten}.$$

PROBLEMS

7. Write "Yes" or "No" to indicate whether each of these is a true statement.

   a. $1_{ten} = 1_{four}$ c. $3_{four} = 3$

   b. $2_{four} = 2_{ten}$ d. $10 = 10_{four}$

8. Express each of these base four numerals as base ten numerals.

   a. 21_{four} b. 30_{four} c. 12_{four}

9. Express each of these base ten numerals as a base four numeral.

   a. 8 b. 14 c. 11

10. Using base four numerals:

    a. Name the even whole numbers less than sixteen.

    b. Name the odd whole numbers not greater than fifteen.

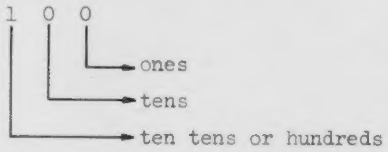
EXTENDING GROUPING BY FOURS

Our base ten numeration system includes more than just two places, a tens place and a ones place. Likewise, a base four numeration system includes more than just a fours place and a ones place. We now consider an extension of grouping by fours.

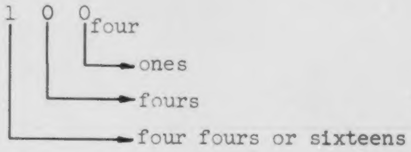
---

*The symbol $\neq$ means "is not equal to".

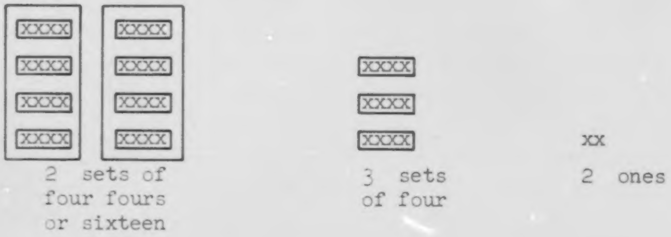
We know that ninety-nine is the greatest whole number that can be names as a two-place numeral in our base ten numeration system: 99. The next whole number, ten tens, or one hundred, necessitates a new place to the left of tens place. Thus, we name ten tens or one hundred with the numeral


1 0 0
ones
tens
ten tens or hundreds

Similarly, fifteen is the last whole number than can be named with a two-place numeral in a base four numeration system: 33. The next whole number, four fours, or sixteen, necessitates a new place to the left of fours place. Thus, we name four fours or sixteen with the numeral


100_{four}
ones
fours
four fours or sixteens

The following diagram may help us interpret a numeral such as 232_{four}.


XXXX
2 sets of four fours or sixteen
3 sets of four
xx
2 ones

Thus, 232_{four} is another name for 46_{ten}: $232_{four} = 46$.

The place values associated with a base four numeration system follow the same pattern as do the place values associated with a base ten numeration system, as shown in this chart:

Base × Base × Base	Base × Base	Base	One
Ten × Ten × Ten (Thousands)	Ten × Ten (Hundreds)	Ten	One
Four × Four × Four (Sixty-fours)	Four × Four (Sixteens)	Four	One

Thus, the numeral 2123_{four} may be interpreted as:

$$2123_{four} = (\underline{2} \times 4 \times 4 \times 4) + (\underline{1} \times 4 \times 4) + (\underline{2} \times 4) + (\underline{3} \times 1)$$

$$2123_{four} = (2 \times 64) + (1 \times 16) + (2 \times 4) + (3 \times 1)$$

$$2123_{four} = 128 + 16 + 8 + 3$$

$$2123_{four} = 155 \quad (\text{i.e., } 155_{ten}) .$$

PROBLEMS

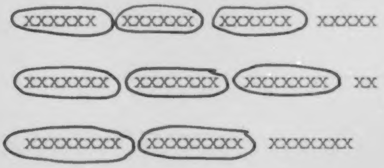
11. Express each of these base four numerals as a base ten numeral.

a. 312_{four} b. 1332_{four} c. 3012_{four}

d. 2301_{four} e. 1230_{four}

OTHER BASES

A set of objects may be grouped in terms of bases other than ten or four. Consider, for instance, a set of 23 objects that are grouped first by sixes, then by sevens, and then by eights.


xxxxxx xxxxxx xxxxxx xxxxx
xxxxxxx xxxxxxx xxxxxxx xx
xxxxxxxx xxxxxxxx xxxxxxx

3 sixes and 5 ones, or 35_{six}

3 sevens and 2 ones, or 32_{seven}

2 eights and 7 ones, or 27_{eight}

These illustrations point to the fact that the place-value pattern associated with base ten and base four may be applied to other bases as well. For instance:

B × B × B × B	B × B × B	B × B	B*	1
10 × 10 × 10 × 10 10000	10 × 10 × 10 1000	10 × 10 100	10 10	1 1
4 × 4 × 4 × 4 256	4 × 4 × 4 64	4 × 4 16	4 4	1 1
3 × 3 × 3 × 3 81	3 × 3 × 3 27	3 × 3 9	3 3	1 1
2 × 2 × 2 × 2 16	2 × 2 × 2 8	2 × 2 4	2 2	1 1
5 × 5 × 5 × 5 625	5 × 5 × 5 125	5 × 5 25	5 5	1 1
6 × 6 × 6 × 6 1296	6 × 6 × 6 216	6 × 6 36	6 6	1 1
7 × 7 × 7 × 7 2401	7 × 7 × 7 343	7 × 7 49	7 7	1 1
8 × 8 × 8 × 8 4096	8 × 8 × 8 512	8 × 8 64	8 8	1 1
9 × 9 × 9 × 9 6561	9 × 9 × 9 729	9 × 9 81	9 9	1 1

*B denotes base.

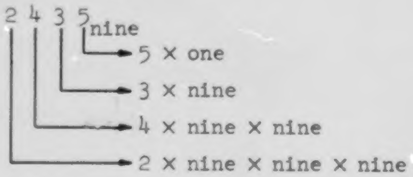
A chart such as the following one may be helpful in showing for the whole numbers one through twenty-five their numerals in each of these bases.

BASE

Ten	Nine	Eight	Seven	Six	Five	Four	Three	Two
1	1	1	1	1	1	1	1	1
2	2	2	2	2	2	2	2	10
3	3	3	3	3	3	3	10	11
4	4	4	4	4	4	10	11	100
5	5	5	5	5	10	11	12	101
6	6	6	6	10	11	12	20	110
7	7	7	10	11	12	13	21	111
8	8	10	11	12	13	20	22	1000
9	10	11	12	13	14	21	100	1001
10	11	12	13	14	20	22	101	1010
11	12	13	14	15	21	23	102	1011
12	13	14	15	20	22	30	110	1100
13	14	15	16	21	23	31	111	1101
14	15	16	20	22	24	32	112	1110
15	16	17	21	23	30	33	120	1111
16	17	20	22	24	31	100	121	10000
17	18	21	23	25	32	101	122	10001
18	20	22	24	30	33	102	200	10010
19	21	23	25	31	34	103	201	10011
20	22	24	26	32	40	110	202	10100
21	23	25	30	33	41	111	210	10101
22	24	26	31	34	42	112	211	10110
23	25	27	32	35	43	113	212	10111
24	26	30	33	40	44	120	220	11000
25	27	31	34	41	100	121	221	11001

As seen from the chart, the base numeral always appears as 10 when written in that particular base system. Similarly, in a particular base system the numeral 100 always designates the number obtained by multiplying the base by itself.

The place-value pattern for a particular base is used whenever we wish to rewrite a numeral in that base as a base ten numeral. Consider, for instance, the place-value pattern applied to the numeral 2435_{nine}:



In terms of this pattern we may write:

$$\begin{aligned} 2435_{\text{nine}} &= (2 \times 9 \times 9 \times 9) + (4 \times 9 \times 9) + (3 \times 9) + (5 \times 1) \\ &= (2 \times 729) + (4 \times 81) + (3 \times 9) + (5 \times 1) \\ &= 1458 + 324 + 27 + 5 \\ &= 1814 \end{aligned}$$

Suppose that we were concerned with the numeral 2435_{six}, instead of the numeral 2435_{nine}. Then, the base six place-value pattern would permit us to write:

$$\begin{aligned} 2435_{\text{six}} &= (2 \times 6 \times 6 \times 6) + (4 \times 6 \times 6) + (3 \times 6) + (5 \times 1) \\ &= (2 \times 216) + (4 \times 36) + (3 \times 6) + (5 \times 1) \\ &= 432 + 144 + 18 + 5 \\ &= 599 \end{aligned}$$

PROBLEMS

12. Express each of these as a base-ten numeral.

a. 3421_{five} b. 5674_{eight} c. 4653_{seven}

d. 20122_{three} e. 32012_{four}

A NOTE ABOUT NOTATION

We have been expressing various nondecimal base numerals as base ten numerals. In this work we moved directly into base ten just as soon as we expressed a nondecimal base numeral in an expanded form. For instance, when we write

$$2134_{\text{five}} = (\underline{2} \times 5 \times 5 \times 5) + (\underline{1} \times 5 \times 5) + (\underline{3} \times 5) + (\underline{4} \times 1)$$

we have expressed all numerals on the right-hand side of the equation in base ten notation.

If for some reason we had wished to express 2134_{five} in an expanded form within base five (rather than in base ten), then we would need to use base five notation throughout the equation. We might convey this idea by writing

$$2134_{\text{five}} = (2 \times 10 \times 10 \times 10)_{\text{five}} + (1 \times 10 \times 10)_{\text{five}} + (3 \times 10)_{\text{five}} + (4 \times 1)_{\text{five}} .$$

These two notations are in keeping with the fact that $5_{\text{ten}} = 10_{\text{five}}$.

On still other occasions an expanded form for 2134_{five} might be expressed as

$$2134_{\text{five}} = (2 \times \text{five} \times \text{five} \times \text{five}) + (1 \times \text{five} \times \text{five}) + (3 \times \text{five}) + (4 \times \text{one}) .$$

In such an instance we have expressed the base consistently as the word "five", thus avoiding the place-value numerals 5_{ten} or 10_{five} .

In practice we select whichever of these forms is best for a particular purpose.

SUMMARY

The main purpose of this chapter has been to assist in developing a deeper understanding of our Hindu-Arabic numeration system, a decimal or base ten system that utilizes a principle of place value. In audition to a consideration of this system itself, attention was directed to two things that hopefully contributed to this deeper understanding: (1) a modified Greek numeration system which had no place-value principle, and (2) place-value numeration systems having bases other than ten.

This latter material should have clarified the fact that the principles which underlie our Hindu-Arabic numeration system are not determined by the fact that its base is ten. These principles are more general ones which can be applied with other bases as well. The case of the decimal base is but a specific illustration of a more general case.

Throughout this chapter we sought to emphasize that any numeration system is a scheme for naming numbers. Although any particular number may be named in various ways, the properties of a number are not affected by the way in which it is named.

APPLICATIONS TO TEACHING

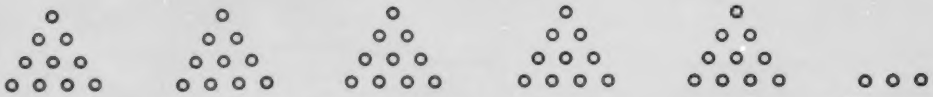
Frequently we display sets of objects in ways that emphasize the *decimal base* of our numeration system. For instance, we may display a set of 53 objects as 5 rows of 10 objects, and 3 more:

o o o o o o o o o o
o o o o o o o o o o
o o o o o o o o o o
o o o o o o o o o o
o o o o o o o o o o
o o o

Representations such as this do help children to think about collections of objects in terms of sets of ten "and some more", and consequently direct attention to the decimal base of our numeration system. This is true of any representation that displays collections of objects as sets of ten, regardless of whether they are arranged in rows, in bundles, or whatever.

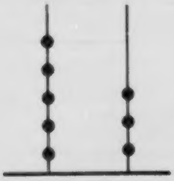
The development of the *place value* concept is a different and more difficult matter. The place value idea is associated with the *numerals* we use, and may or may not be reflected in the way in which a set of objects is arranged.

In the numeral 53 the 5 is in tens place and the 3 is in ones place. However, when a set of 53 objects is displayed in rows of ten (and some ones), as above, the display itself does not suggest the idea of a tens place and a ones place in our numeral system. But we may move in the direction of this idea by showing a collection of 53 objects in such a way that sets of 10 are placed at the left of the ones.



With some objects we often show each set of ten as a "bundle" rather than as shown above. In either instance, we show the sets of ten to the left of the ones, "hinting" at the place value idea associated with numerals. We often further this "hinting" by using place value devices in which sets of ten or bundles of ten are placed in "pockets" marked TENS, and remaining single objects are placed in "pockets" marked ONES.

An abacus representation of 53 clearly is associated much more closely with the place value principle.



Here the number of tens and the number of ones are shown by the beads in different positions. The number of tens and the number of ones also may be shown by tally marks (as at the left below) or by digits (as at the right below) in appropriate positions.

Tens	Ones

Tens	Ones
5	3

We should be aware of the different purposes and uses that are associated with two forms of number charts:

Counting Chart

1	2	3	4	5	6	7	8	9	10
11	12	13	14	15	16	17	18	19	20
21	22	23	24	25	26	27	28	29	30
31	32	33	34	35	36	37	38	39	40
41	42	43	44	45	46	47	48	49	50

Numeral Chart

0	1	2	3	4	5	6	7	8	9
10	11	12	13	14	15	16	17	18	19
20	21	22	23	24	25	26	27	28	29
30	31	32	33	34	35	36	37	38	39
40	41	42	43	44	45	46	47	48	49

The Counting Chart highlights ten as the base of our numeration system. If we locate 35 , for instance, on the Counting Chart, it clearly may be associated with 3 rows of 10 "blocks" and 5 "blocks" in the next row.

The Numeral Chart highlights an important feature of the structure of our numerals. The first row of numerals lists the ten basic symbols or digits used in our numeration system. The second row of numerals includes those with 1 in tens place; the third row, those with 2 in tens place, etc.

Each chart has its appropriate place in an instructional program.

If a child is able to complete correctly an example such as

47 = ___ tens + ___ ones

this does not guarantee that he also can complete correctly an example such as

4 tens + 7 ones = ___.

The development of an understanding of the place value principle demands that children explore its meaning and application with a variety of representations and in a variety of ways. Suggestions made here regarding numbers less than one hundred can be extended, of course, to apply to numbers greater than ninety-nine.

QUESTION

"Why do we have to teach other base numeration systems?"

Contrary to the belief of many lay persons, the reason for teaching other base numeration systems such as base-two numeration is not due to the increasing influence of computers in our society. The main intention is to sharpen our understanding of our own decimal numeration system. It is not the ultimate goal that the children should be able to compute with facility in other numeration systems. In the attempt to understand how and why the decimal system works, often rote computation dulls our sense to what occurs in the computation by the mere fact of being too familiar with the mechanics. By forcing ourselves to see the grouping and regrouping that may be necessary in other bases, for example, we see more keenly the rationale behind so-called "carrying" and "borrowing". To illustrate, the traditional method of dealing with finding the combined length of an object 1 foot 8 inches and another 1 foot 7 inches is presented:

$$\begin{array}{rlrl} 1 & \text{foot} & 8 & \text{inches} \\ 1 & \text{foot} & 7 & \text{inches} \\ \hline 2 & \text{feet} & 15 & \text{inches} \end{array}$$

One may shudder to find youngsters "carrying" without regard to what is being carried:

$$\begin{array}{rlrl} 1 & \text{foot} & 8 & \text{inches} \\ 1 & \text{foot} & 7 & \text{inches} \\ \hline 3 & \text{feet} & 5 & \text{inches} \end{array}$$

Here, it is clear to us that twelve of the units called "inches" are required for regrouping into each foot, and the combined length should be 2 feet 15 inches, or, regrouping, 3 feet 3 inches. The associated computation is in base twelve; $18_{\text{twelve}} + 17_{\text{twelve}} = 33_{\text{twelve}}$.

VOCABULARY

Base (of a numeration system)*
Base-Four System
Counting Chart
Decimal System*
Digit*
Numeral*
Numeral Chart
Numeration System*
Place Value*

EXERCISES - CHAPTER 6

1. In each ring write = or > or < so that the sentence will be true.
   a. $7000 + 600 + 50 \bigcirc 7000 + 60 + 5$
   b. $(3 \times 1000) + (8 \times 100) + (4 \times 1) \bigcirc 3840$
   c. $1234_{\text{eight}} \bigcirc 1234$
   d. $4321_{\text{six}} \bigcirc 4321_{\text{five}}$
   e. $400 \bigcirc 3100_{\text{five}}$
   f. $3120_{\text{four}} \bigcirc (3 \times 4 \times 4 \times 4) + (1 \times 4 \times 4) + 2 \times 1)$
2. Express each of the following as base-ten (decimal) numerals.
   a. 13_{five}
   b. 24_{eight}
   c. 123_{five}
   d. 103_{six}
   e. 72_{nine}
   f. 1111_{five}

SOLUTIONS FOR PROBLEMS

1. a. 47 b. 852 c. 620 d. 403 e. 999
2. a. OC b. YLE c. TJ d. WD e. ZQH
3. No. It is not needed since the system has no place-value principle.
4. a. 2747 b. 81330 c. 460093
5. a. 749 b. 8306 c. 2751 d. 46083
   e. 5024 f. 70609 g. 8403 h. 95060

6. a. 6184: $6000 + 100 + 80 + 4$
$(6 \times 1000) + (1 \times 1000) + (8 \times 10) + (4 \times 1)$
$(6 \times 10 \times 10 \times 10) + (1 \times 10 \times 10) + (8 \times 10) + (4 \times 1)$

b. 7350: $7000 + 300 + 50$
$(7 \times 1000) + (3 \times 100) + (5 \times 10)$
$(7 \times 10 \times 10 \times 10) + (3 \times 10 \times 10) + (5 \times 10)$

c. 40702: $40000 + 700 + 2$
$(4 \times 10000) + (7 \times 100) + (2 \times 1)$
$(4 \times 10 \times 10 \times 10 \times 10) + (7 \times 10 \times 10) + (2 \times 1)$

7. a. Yes b. Yes c. Yes d. No $(4 = 10_{\text{four}})$

8. a. 9 b. 12 c. 6

9. a. 20_{four} b. 32_{four} c. 23_{four}

10. a. 0_{four}, 2_{four}, 10_{four}, 12_{four}, 20_{four}, 22_{four}, 30_{four}, 32_{four}.
b. 1_{four}, 3_{four}, 11_{four}, 13_{four}, 21_{four}, 23_{four}, 31_{four}, 33_{four}.

11. a. 54 b. 126 c. 198 d. 177 e. 108

12. a. $(3 \times 125) + (4 \times 25) + (2 \times 5) + (1 \times 1) = 486$
b. $(5 \times 512) + (6 \times 64) + (7 \times 8) + (4 \times 1) = 3004$
c. $(4 \times 343) + (6 \times 49) + (5 \times 7) + (3 \times 1) = 1704$
d. $(2 \times 81) + (1 \times 9) + (2 \times 3) + (2 \times 1) = 179$
e. $(3 \times 256) + (2 \times 64) + (1 \times 4) + (2 \times 1) = 902$

CHAPTER 7

ADDITION

OPERATIONS

The four basic operations of arithmetic are addition, subtraction, multiplication and division. A binary operation in mathematics is a way of associating with an ordered pair of numbers a unique third number. An ordered pair is a set of two objects, not necessarily different, one of which is designated as the first object of the pair. If dog is the first element of an ordered pair and cat is the second element, we usually write (dog, cat) to indicate the ordered pair.

When we are performing the operation of addition, we associate the number 8 with the ordered pair (6, 2). When we are performing the operation of subtraction, we associate 4 with this same ordered pair, (6, 2). For the operation of multiplication, 12 is associated with (6, 2) and for division, 3 is associated with (6, 2).

UNION OF SETS AND ADDITION

The union of disjoint sets is the basis for the concept of adding whole numbers.

If

$$A = \{a, b, c, d, e\}$$

and

$$B = \{x, y, z\},$$

then

$$A \cup B = \{a, b, c, d, e, x, y, z\}.$$

We know that $N(A) = 5$, $N(B) = 3$ and $N(A \cup B) = 5 + 3$, or 8.

The sum of the cardinal numbers of two disjoint sets is defined as the cardinal number of the union of the two sets.

We say

$$3 + 5 = 8.$$

Three and 5 are called addends, 8 is the sum.

When we start with two disjoint sets and form the union, we are operating on sets. When we start with two numbers and get a third we are operating on numbers. Addition is a binary operation on the cardinal numbers associated with two disjoint sets.

We call addition a binary operation because we operate on just two numbers at a time. Union is an operation on sets. Addition is an operation on numbers. We join (form their union) sets and we add numbers.

PROPERTIES UNDER ADDITION

Since addition is associated with the union of sets, we can expect that properties under the union operation may have implications for the addition operation. We observe first, that the union of two sets is a set. This, of course, is from the definition of union. As a whole number may be assigned to any finite set, corresponding to the fact that

THE UNION OF TWO SETS IS A SET,

we have

THE SUM OF TWO WHOLE NUMBERS IS A WHOLE NUMBER.

Both of these are statements of closure properties. The first is the closure property of sets under union, and the second is the closure property of whole numbers under addition. If an operation that is defined on a set is such that the result is an element of the same set, then we say that the set is closed under the operation. For example, let us consider the set of whole numbers

$$W = \{0, 1, 2, \dots\}$$

and the operation described by "doubling the number".

Then we see

the doubling of 1 gives 2,
the doubling of 2 gives 4,
the doubling of 3 gives 6, and so on.

In other words if we perform this operation on any whole number we get its double. That is, the result of performing the operation on any whole number n is $2 \times n$. Since doubling any whole number is again a whole number we say the set of whole numbers is closed under the operation of doubling.

When we add any two whole numbers the result is always a whole number. This means that every time we add two whole numbers, the result is always in the set of whole numbers. A consequence of this property is that we may repeat the operation using the sum as one of the addends.

Another property of sets under union pertains to the order of operation. If A and B are sets, the result of joining A to B is the same as joining B to A. We summarize this by saying that the union is a commutative operation. For any sets A and B,

$$A \cup B = B \cup A.$$

Corresponding to this, we have the commutative property of whole numbers under addition. For any whole numbers a and b,

$$a + b = b + a.$$

For instance, the sum of 3 and 4 (which may be written $3 + 4$) and the sum of 4 and 3 (written $4 + 3$) both are the same number, 7. For this reason, we can write

$$3 + 4 = 4 + 3.$$

Both $3 + 4$ and $4 + 3$ name the same number.

We have said above that a consequence of the closure property under addition is that the operation may be repeated on the sum. For example, since $3 + 4$ is a whole number we might add another whole number say, 9, to the sum. This would be indicated in the grouping of $3 + 4$, in parentheses, thus:

$$(3 + 4) + 9.$$

Since the sum $3 + 4$ is also 7, the expression $(3 + 4) + 9$ means the sum $7 + 9$; or, in other words, 16. That is to say,

$$(3 + 4) + 9 = 7 + 9, \text{ and } 7 + 9 = 16;$$

therefore, $(3 + 4) + 9 = 7 + 9$

$$= 16.$$

Since 16 is a whole number, this process may be continued as needed. Thus, we may add say, 5, to the result of $(3 + 4) + 9$ to get the result of $((3 + 4) + 9) + 5$, which is the same as $16 + 5$, or 21.

Our next concern is to pursue the concept of grouping the addends. Recall that for sets, the grouping under union did not change the resulting set. That is, union is said to be an associative operation. Consequently, both $(A \cup B) \cup C$ and $A \cup (B \cup C)$ give rise to the same number property. Therefore, we have the associative property of whole numbers under addition:

FOR WHOLE NUMBERS a, b, AND c,

$$(a + b) + c = a + (b + c).$$

If A has the number property 3, B has the number property 4, and C has the number property 9, then $A \cup B$ has the number property 7 and $(A \cup B) \cup C$ has the number property $7 + 9$, or 16. For these same sets $B \cup C$ has the number property 13 and $A \cup (B \cup C)$ has the number property $3 + 13$, or 16. A, B, C are of course, all disjoint since addition is derived from the union of disjoint sets. To trace "the machinery" behind this property, we can display $(a + b) + c$ and $a + (b + c)$ as follows:

$$\begin{array}{ccc} (3 + 4) + 9 & & 3 + (4 + 9) \\ \shortparallel & & \shortparallel \\ 7 + 9 & & 3 + 13 \\ \shortparallel & & \shortparallel \\ 16 & = & 16 \end{array}$$

with the vertical equal signs indicating equality as we read vertically. This may be interpreted as follows:

$$(3 + 4) + 9 = 7 + 9 = 16;$$

independently $\quad 3 + (4 + 9) = 3 + 13 = 16.$

Since $16 = 16$, we can follow the chain thus:

$$(3 + 4) + 9 \longrightarrow 7 + 9 \rightarrow 16 \rightarrow 16 \rightarrow 3 + 13 \rightarrow 3 + (4 + 9).$$

From this, we conclude that $(3 + 4) + 9 = 3 + (4 + 9)$. The associative property states that this characteristic is not restricted to just the numbers 3, 4 and 9; it holds for any whole numbers a, b, and c; that is, $(a + b) + c = a + (b + c)$.

The property for closure allows us to repeatedly add as many numbers as we wish. The commutative and associative properties allow us to do the adding in whichever way we please, as long as each addend is appropriately accounted for. For example, we may require the sum:

$$3 + 6 + 9 + 4 + 7.$$

Closure states that this can be done; merely add any two, then continue to add any of the other addends to the result and so on. Commutativity and associativity say that if we so choose, we are free to pick appropriate combinations at will.

For instance, in the above example, it may be desirable to look for combinations of ten since adding one ten to another is easy for us. For the above sum, we may then find it convenient to group in the following way: $(6 + 4)$, $(3 + 7)$. Hence, the scheme of our procedure and the reasons permitting us to use this scheme is:

$$\begin{aligned}
3 + 6 + 9 + 4 + 7 &= (3 + 6) + (9 + 4) + 7 \\
&= (3 + 6) + (4 + 9) + 7 && \text{Commutative Property} \\
&= 3 + [6 + (4 + 9)] + 7 && \text{Associative Property} \\
&= 3 + [(6 + 4) + 9] + 7 && \text{Associative Property} \\
&= 3 + 7 + [(6 + 4) + 9] && \text{Commutative Property} \\
&= (3 + 7) + (6 + 4) + 9 && \text{Associative Property}
\end{aligned}$$

The use of the commutative and associative properties of addition allows us to go leap-frogging and add any two pairs of numbers we choose in finding the sum of many numbers. Use of the commutative and associative properties also is the basis for checking addition by adding in the opposite direction. For example, to add

$$\begin{array}{r} 3 \\ 6 \\ 4 \\ 8 \\ \hline \end{array} \uparrow \downarrow$$

we might add up and have

$$[(8 + 4) + 6] + 3.$$

If we add down we have

$$[(3 + 6) + 4] + 8.$$

We can show that these two name the same number because of the commutative and associative properties.

$[(8 + 4) + 6] + 3 = (8 + 4) + (6 + 3)$ Associative Property
$= (8 + 4) + (3 + 6)$ Commutative Property
$= (3 + 6) + (8 + 4)$ Commutative Property
$= (3 + 6) + (4 + 8)$ Commutative Property
$= [(3 + 6) + 4] + 8$ Associative Property

PROBLEMS*

1. Which of the following statements are examples of the commutative property under addition?
   a. $7 + 8 = 8 + 7$
   b. $9 + 8 = 8 + 9$
   c. $(7 + 8) + 9 = (8 + 7) + 9$
   d. $(7 + 8) + 9 = 7 + (8 + 9)$
   e. $78 = 87$
   f. $(7 + 8) + 9 = 9 + (7 + 8)$
   g. $7 + 8 + 9 = 9 + 8 + 7$

2. Which of the following statements are examples of the associative property under addition?
   a. $(7 + 8) + 9 = (7 + 8) + 9$
   b. $(7 + 8) + 9 = 7 + (8 + 9)$
   c. $(7 + 8) + 9 = 9 + (7 + 8)$
   d. $7 + 8 + 9 = (7 + 8) + 9$
   e. $7 + 8 + 9 + 10 = (7 + 8) + (9 + 10)$
   f. $(7 + (8 + 9)) + 10 = ((8 + 9) + 7) + 10$
   g. $(7 + (8 + 9)) + 10 = 7 + ((8 + 9) + 10)$

3. Which property or properties of whole numbers under addition make(s) each of the following true?
   a. $(7 + 8) + (9 + 10) = (9 + 10) + (7 + 8)$
   b. $(7 + 8) + (9 + 10) = (7 + 8) + (10 + 9)$
   c. $7 + 8 = 15$
   d. $7 + 8 + 9 + 10 = 10 + 9 + 8 + 7$
   e. $7\ 8\ 9 = 9\ 8\ 7$
   f. $7 + (8 + 9) + 10 = (7 + 8) + (9 + 10)$
   g. $7 + 8 + 9 + 10 = (7 + 10) + (8 + 9)$

---

* Solutions to problems in this chapter will be found on page 128.

Another property of sets under the union operation that is significant for the addition operation is one that is connected with the union of a set with the empty set. We have observed before that if A is a set then A ∪ { } = A. Since the number property of the empty set is 0, if the number property of A is a, then the corresponding statement for the above observation is:

FOR ANY WHOLE NUMBER a,

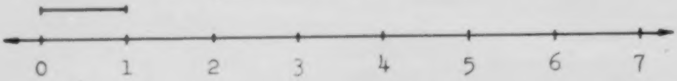
$$a + 0 = a.$$

Of course, because of the commutative property, we also have $0 + a = a$.

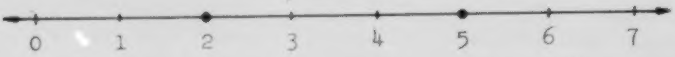
Since addition of 0 to any number produces that identical number, 0 is called the identity element with respect to addition. No other element plays this same role. The property referred to above is known as the property of zero under addition, or in short, the addition property of zero.

ADDITION ON THE NUMBER LINE

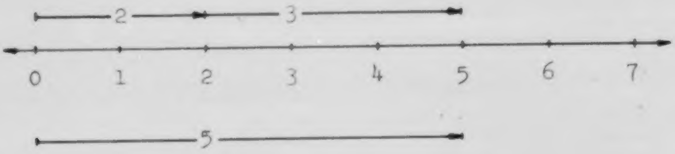
The operation of addition may be vividly pictured on the number line. Recall that the number line is constructed by placing marks on a line so that the segment between any two neighboring marks is congruent to one chosen segment. This was accomplished by laying off copies of the chosen segment end to end. The chosen segment determines a unit in the number line.



To visualize $2 + 3 = 5$, let us first locate 2 and 5 on the number line; notice that between 2 and 5 are 3 units. Furthermore, we can observe that between 0 and 2 are 2 units.



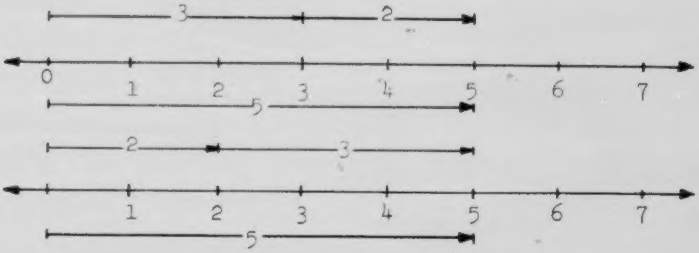
This process may be more effectively indicated by arrows as illustrated below, showing $2 + 3 = 5$.



$2 + 3 = 5$

The above diagram shows an addition using the number line. More than this, however, the example may be interpreted also as an illustration of the closure property. An arrow of 2 units "followed by" an arrow of 3 units is associated with an arrow of a whole number of units. Each unit may be regarded as a step. Thus, 2 steps followed by 3 steps result in a total of 5 steps. Note that the steps originate from 0 as starting point and that we advance in accord with the increasing order of numbers.

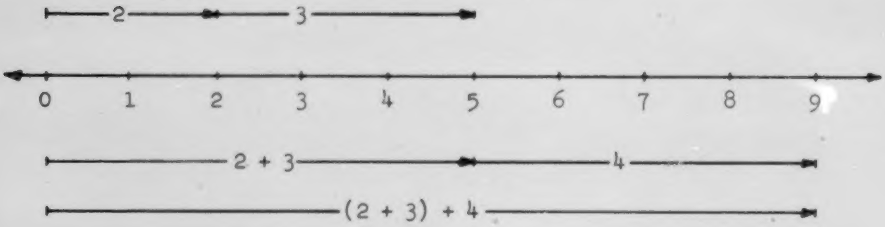
Consider now the sum 3 + 2 on the number line. Here, 3 steps are followed by 2 steps and it is clear that we get the same result as before. Incorporating the diagrams for 3 + 2 = 5 and 2 + 3 = 5 into a single diagram, we can illustrate the commutative property under addition.



The associative property can also be illustrated using the number line. However, the process is more involved. As an example, we know that

$$(2 + 3) + 4 = 2 + (3 + 4).$$

The first expression, $(2 + 3) + 4$, may be illustrated by a simple extension of the above method. An arrow of 5 units results from the 2 and 3 unit arrows. To this, is abutted (attached end to end) the 4 unit arrow, thus



(2 + 3) + 4 = 9

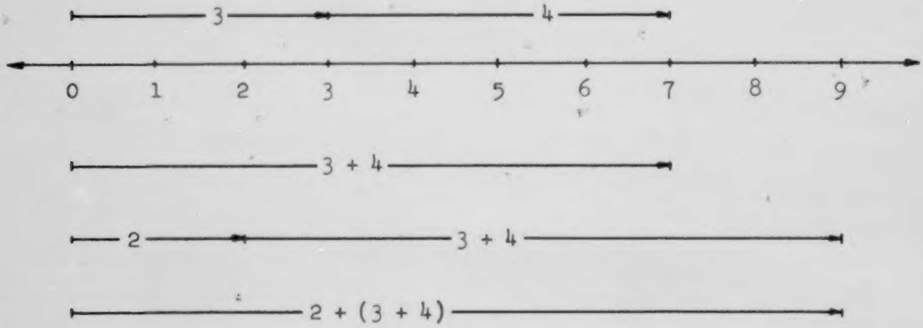
This of course, is analogous to the chain of statements

$$(2 + 3) + 4 = 5 + 4 = 9.$$

The illustration for the second expression, $2 + (3 + 4)$, is not as direct. For this, it may be more helpful to start with the analogous situation first. In analyzing $2 + (3 + 4)$, we note that $3 + 4 = 7$; that is " 3 + 4" and "7" are names for the same number. Thus,

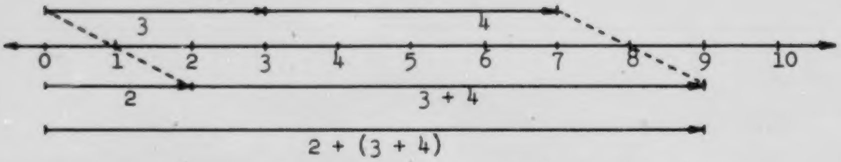
$$2 + (3 + 4) = 2 + 7 = 9.$$

Accordingly, we are seeking an arrow corresponding to $3 + 4$. This arrow is then abutted to the arrow of 2 units to arrive at the result for $2 + (3 + 4)$.

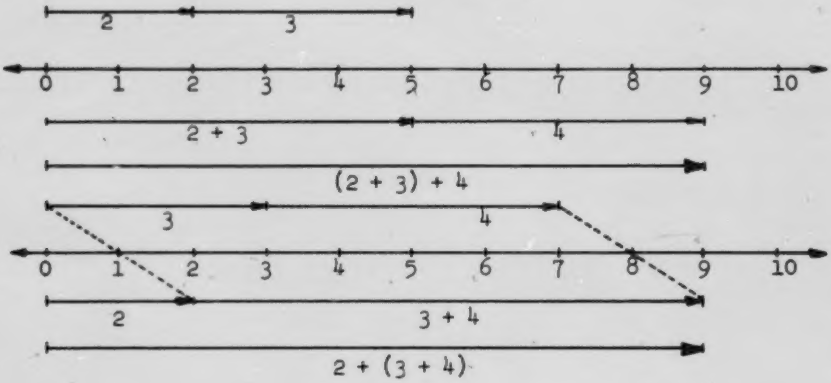


2 + (3 + 4)= 9

The diagramming may be simplified by transferring the arrow for 3 + 4 directly onto the 2 unit arrow as is shown below by the dotted lines:


3
4
0
1
2
3
4
5
6
7
8
9
10
2
3 + 4
2 + (3 + 4)

It is by incorporating the diagrams for $(2 + 3) + 4 = 9$ and for $2 + (3 + 4) = 9$ that we show associativity.


2
3
0
1
2
3
4
5
6
7
8
9
10
2 + 3
4
(2 + 3) + 4
3
4
0
1
2
3
4
5
6
7
8
9
10
2
3 + 4
2 + (3 + 4)

Frequent use of the number line to illustrate addition of whole numbers will promote familiarity with properties under addition. Thus the number line can help a great deal in working with numbers and in answering questions about numbers.

PROBLEMS

4. Draw number lines to show the following addition examples.
   a. $3 + 6 = 9$
   b. $4 + 5 = 9$
   c. $(3 + 6) + 7 = 16$
   d. $3 + (6 + 7) = 16$

5. Draw number lines to show that the following numbers are commutative under addition.
   a. 3 and 5
   b. 30 and 50
   c. $(3 + 6)$ and 7

6. Are the diagrams in Problem 5c the same as those in Problems 4c and 4d? Why or why not?

7. How would arrows be used to indicate advancing from one point on the whole number line to the next point? What does this suggest about the whole number immediately following a given whole number a?

SUMMARY OF PROPERTIES

The properties of addition developed so far for whole numbers may be summarized as follows, where a, b, and c are whole numbers

1. The set of whole numbers is closed under addition.

   $a + b$ is a whole number

2. Addition of whole numbers is a commutative operation.

   $$a + b = b + a$$

3. Addition of whole numbers is an associative operation.

   $$(a + b) + c = a + (b + c)$$

4. There is an identity element 0 for addition

   $$a + 0 = 0 + a = a.$$

NUMBER SENTENCES

In developing the properties of numbers and various operations on numbers, we have been using a rather special language involving:

Symbols for numbers, such as: 1, 5, 2, 9, 3, ...;
Symbols for operations, such as: +, ×;
and Symbols showing relations between numbers,
such as: =, >, <.

A great deal of mathematics is in the form of sentences about numbers or number sentences as they are called. Sometimes the sentences make true statements as in "$9 + 5 = 14$", sometimes the number sentences are false as in "$5 + 7 = 11$". Whether it is true or false no more disqualifies the statement as a sentence than the statement, "George Washington was vice president under Abraham Lincoln" is disqualified as a sentence.

Any number sentence has to have a "verb" or "verb form". The ones we have encountered so far are: "is equal to", "is less than", "is greater than". The symbols which we use for these verbs are listed below with a number sentence illustrating the use of each.

$=$; "is equal to"; $3 + 4 = 7$

$<$; "is less than"; $5 < 2 \times 5$

$>$; "is greater than"; $7 + 1 > 7$

As we have noted, verbal sentences may be true: "George Washington was the first President of the United States," or false: "Abraham Lincoln was the first President of the United States." We also encounter sentences such as: "He was the first President of the United States." If read out of context, it may not be known to whom "he" referred and it may thus be impossible to determine whether the sentence is true or false. In fact, " ☐ was the first President of the United States" may be a test question requiring the name of the man for which it would be a true sentence. Such a sentence is called an open sentence and is of great usefulness not only in history tests but in many other situations as well. Open number sentences are the basis of a great deal of work in arithmetic. Solving a problem in arithmetic, for example, incorporates the notion of an open sentence. As an illustration, the problem

$\begin{array}{r} 7 \\ +\ 5 \\ \hline \end{array}$ may be stated: $7 + 5 = \square$ or $7 + 5 = \underline{\quad}$.

The number that makes $7 + 5 = \square$ a true statement is the solution for

$$\begin{array}{r} 7 \\ +\ 5 \\ \hline \end{array}.$$

Open number sentences are called equations if the verb in them is "=". Sentences with any of the other verbs listed above are called "inequalities".

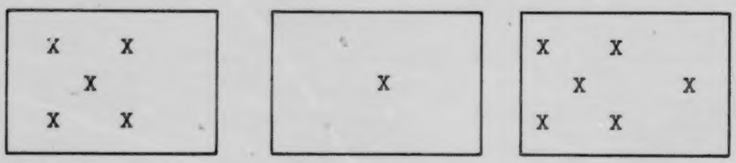
PROBLEM

8. Write <, >, or = in each blank so each mathematical sentence is true.

   a. 8 ____ 6
   b. 3 + 4 ____ 16
   c. (20 + 30) ____ (30 + 20)
   d. (200 + 800) ____ (200 + 700)
   e. (1200 + 1000) ____ (1000 + 1200)

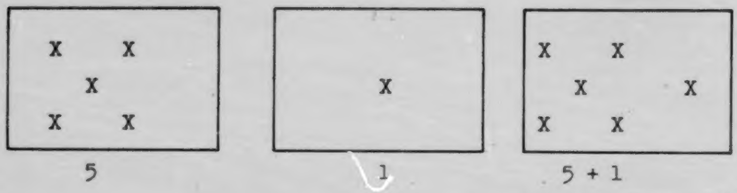
APPLICATIONS TO TEACHING

Addition is associated with the union of disjoint sets. By this, the commutative property is clearly illustrated; whether we join the first set to the second set or the second set to the first, the union consists of the same members. Recording results of joining sets using numerals may cause some difficulty without some intermediate steps. For example, from the diagram

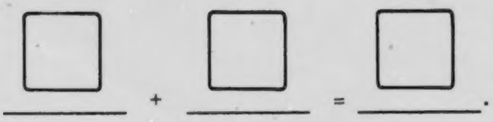


some children might not be able to proceed directly to the number sentence, 5 + 1 = 6.

A suggestion is to separate this problem into different tasks. Use of the flannel board to display objects in each set will be helpful. Then the numerals may be written below each picture with the numeral for the union showing the addends.

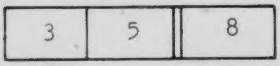


This may be followed by a review of the procedure the next day, writing 6 below 5 + 1 and finally, completion of the equation ____ + _____ = ____. Some teachers have reported considerable success with providing each child a specially outfitted cigar box for this task. The lid of the box is lined with some flannel material on which three frames have been drawn. Beneath these frames appear the "skeleton" sentence, _____ + _____ = _____. That is, the personalized flannel board looks something like this:



With each problem, the child constructs sets with color paper cutouts that he has in the box, and completes the corresponding number sentence with construction paper cutouts on which have been written various numerals.

In forming their own sentences to accompany a pictorial situation, some children may have difficulty getting the "=" symbol in the right place. Drawing a double line between the appropriate frames may help with the association of ideas.



The use of the number line has been reported to be extremely helpful. A number line is fastened to each child's desk; the child eventually operates independent of this device in accord with his own rate of development.

QUESTION

"Why do we say that an operation is a way of associating an ordered pair of numbers with a unique third number? Isn't it true that both (6,2) and (2, 6) result in 8 for addition?"

It is true that both (6, 2) and (2, 6) result in the same number under addition. This is the property which we call the commutative property of whole numbers under addition, illustrated here by

$$6 + 2 = 2 + 6.$$

The necessity for considering an operation as associated with an ordered pair is more sharply brought to the fore when the operation is not commutative, as for example, in subtraction, $6 - 2$ and $2 - 6$ do not have the same result, and it is a vital issue which of the two numbers is considered the first number and which the second under the operation.

VOCABULARY

Addend*
Addition*
Associative Property of Addition*
Closure Property of Whole Numbers under Addition*
Commutative Property of Addition*
Equation*
Identity Element*
Inequalities
Number Sentences
Ordered Pair
Sum*

EXERCISES - CHAPTER 7

1. If the operation of addition is applied to each ordered pair below, what whole number is associated to each ordered pair?
   a. $(3, 4)$
   b. $(9, 8)$
   c. $(16, 7)$
   d. $(24, 36)$
   e. $(36, 24)$
   f. $(7, 16)$
2. Which ordered pairs in Exercise 1 give the same number? Why?
3. Which of the following sentences are true for any whole numbers a and b? Why?
   a. $(a + b) + 0 = a + b$
   b. $(a + b) + 9 = a + (b + 9)$
   c. $(a + b) + c = (b + a) + c$
4. By inspection give a whole number that makes each sentence below true.
   a. $3 + \square = 10$
   b. $7 + 16 = \square$
   c. $\square + 8 = 14$
   d. $\square + 99 = 500$

5. What properties of addition of whole numbers are illustrated by each of the following statements?

   a. 5 + 7 = 7 + 5

   b. 3 + 0 = 3

   c. 8 + (6 + 4) = (8 + 6) + 4

   d. 8 + (9 + 7) = 8 + (7 + 9)

   e. 0 + 18 = 18

   f. 14 + (9 + 7) = (9 + 7) + 14

SOLUTIONS FOR PROBLEMS

1. a, b, c, f

2. b, d, e, g

3. a. commutative property

   b. commutative property

   c. closure property

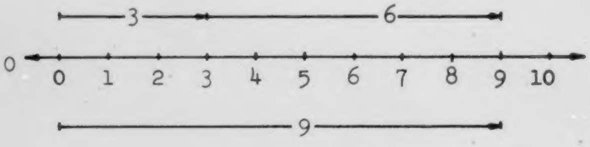
   d. commutative property

   e. no property; statement is false

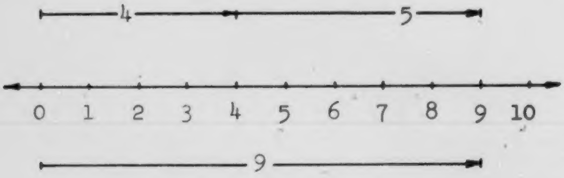
   f. associative property

   g. commutative and associative property.

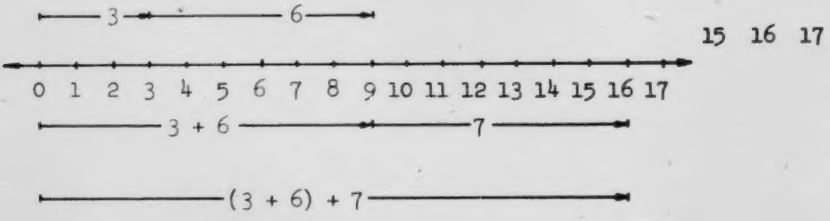
4. a.


3
6
0
0 1 2 3 4 5 6 7 8 9 10
9

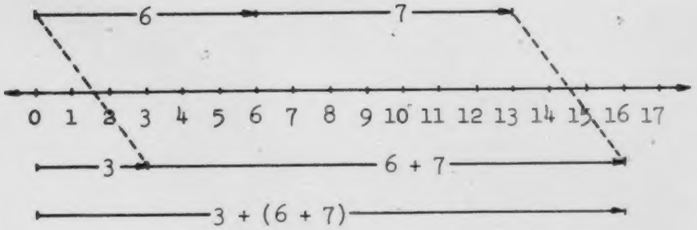
b.


4
5
0 1 2 3 4 5 6 7 8 9 10
9

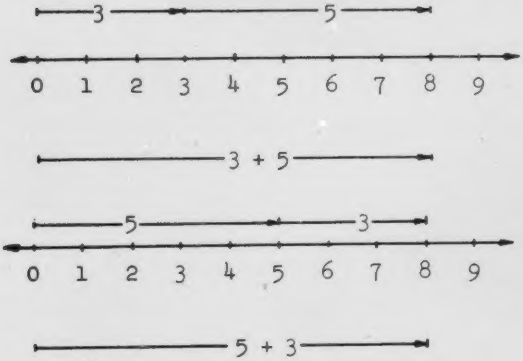
c.


3
6
15 16 17
0 1 2 3 4 5 6 7 8 9 10 11 12 13 14 15 16 17
3 + 6
7
(3 + 6) + 7

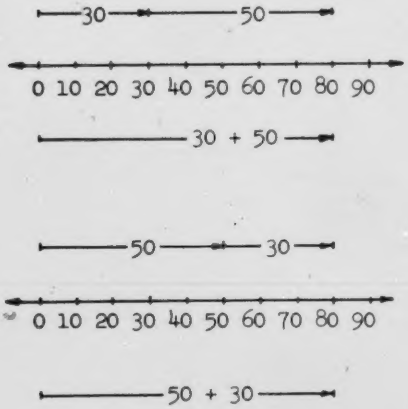
d.


6
7
0 1 2 3 4 5 6 7 8 9 10 11 12 13 14 15 16 17
3
6 + 7
3 + (6 + 7)

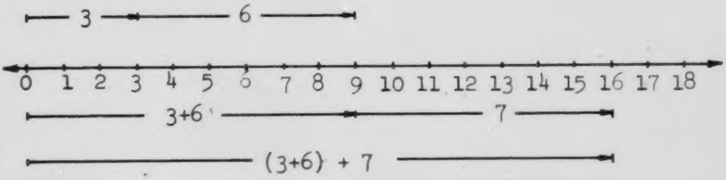
5. a.

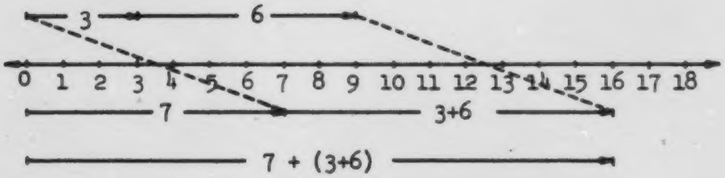

3
5
0 1 2 3 4 5 6 7 8 9
3 + 5
5
3
0 1 2 3 4 5 6 7 8 9
5 + 3

b.


30
50
0 10 20 30 40 50 60 70 80 90
30 + 50
50
30
0 10 20 30 40 50 60 70 80 90
50 + 30

c.


3
6
0 1 2 3 4 5 6 7 8 9 10 11 12 13 14 15 16 17 18
3+6
7
(3+6) + 7


3
6
0 1 2 3 4 5 6 7 8 9 10 11 12 13 14 15 16 17 18
7
3+6
7 + (3+6)

6. No; 4c and 4a show associativity of $3 + 6 + 7$; 5c shows commutativity of $(3 + 6)$ and 7.

7. Abutting a 1 unit arrow to an arrow corresponding to a given number. This shows that the whole number after a given whole number a is obtained by adding 1 to a.

Chapter 8

MULTIPLICATION

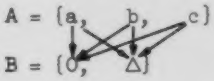
MULTIPLICATION AND THE PRODUCT SET

Multiplication of whole numbers is a binary operation which associates with two whole numbers called factors and a unique third whole number called the product.

Multiplication is related to the product set of two sets just as addition is related to the union of two disjoint sets. The product set of two sets, A and B, results from a process of pairing each element of set A with each element of set B. For example let

$$A = \{a, b, c\}$$
$$B = \{0, \triangle\}$$

We find the product set $A \times B$ by pairing each element of A with each element of B as shown below



This is the set of ordered pairs, $A \times B = \{(a, 0), (a, \triangle), (b, 0), (b, \triangle), (c, 0), (c, \triangle)\}$. An orderly arrangement of these pairings is called an array. There are 6 different pairs, three rows and two columns as shown on the left below

	0	△
a.	(a, 0)	(a, △)
b.	(b, 0)	(b, △)
c.	(c, 0)	(c, △)

6 different pairs

$$\begin{array}{c|cc} & \multicolumn{2}{c}{2} \\ \hline & 0 & 0 \\ 3 & 0 & 0 \\ & 0 & 0 \end{array}$$

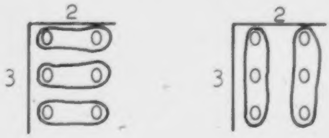
$3 \times 2 = 6$

The number property of $A \times B$ is

$$N(A \times B) = 3 \times 2, \text{ or } 6.$$

This cardinal number is called the product of 3 and 2. The product, written 3 × 2, and read "three times two" is defined as the number property of $A \times B$ when $N(A) = 3$ and $N(B) = 2$. The array of $A \times B$ may be drawn as shown on the right above. The product is the number of dots in the array.

Notice that a 3 × 2 array may be viewed as the union of three disjoint sets each having 2 elements. In



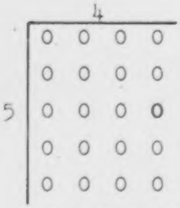
other words, 3 × 2 may be thought of as the sum of three twos,

$$3 \times 2 = 2 + 2 + 2.$$

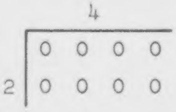
The product 3 × 2 may also be thought of as the union of two disjoint sets each having three elements. Thus, 3 × 2 may be thought of as the sum of two threes,

$$3 \times 2 = 3 + 3.$$

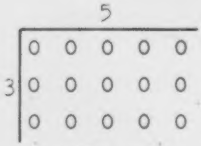
Every product involving counting numbers may be represented by an array. Some arrays are shown below.



$5 \times 4 = 20$



$2 \times 4 = 8$



$3 \times 5 = 15$

On the basis of such arrays, we can think of multiplication in terms of counting sets as follows:

> GIVEN NUMBERS a AND b, AN a BY b RECTANGULAR ARRAY OF OBJECTS CAN BE CONSTRUCTED SUCH THAT THERE ARE a ROWS AND b COLUMNS IN THE ARRAY. THE NUMBER, a × b, IS THE NUMBER OF OBJECTS IN THE ARRAY.

PROBLEMS*

1. Draw arrays illustrating the following products.

a. 3×5	d. 7×3
b. 6×2	e. 8×5
c. 2×6	f. 9×4

2. Given $A = \{a, b\}$, $B = \{0, \triangle, \square, \boxtimes\}$. Find $A \times B$. Find $B \times A$. What is $N(A \times B)$? What is $N(B \times A)$? Is $A \times B = B \times A$? Is $N(A \times B) = N(B \times A)$?

PROPERTIES UNDER MULTIPLICATION

In the above, we have related multiplication to the product set. The result of the operation of multiplication on any pair of numbers is called the product of the two numbers.

When we examined the union of two sets to get an insight into the properties under addition, we observed that the union of the two sets is a set. The product set may similarly be examined to gather some information on the properties of the set of whole numbers under multiplication. As in the case of union, the product set of two sets is also a set. It is true that the elements of the product set are not elements of the original sets--they are ordered pairs of these elements. But, the crucial point is that the cartesian product is a set, and a number property may be assigned to this set. From this, we can intuitively accept the closure property of whole numbers under multiplication:

THE PRODUCT OF TWO NUMBERS IS A WHOLE NUMBER.

If $A = \{a, b, c, d\}$ and $B = \{\alpha, \beta, \gamma, \delta, \epsilon\}$, then the product set $A \times B$ is a set with 20 members. We have seen that if $A \neq B$, then the cartesian product $B \times A$ is different from $A \times B$ since the pairs are ordered. For example, (a, β) is a member of $A \times B$ whereas (β, a) is a member of $B \times A$. By displaying the members of $B \times A$ as we had done for $A \times B$ we should see that $B \times A$ also has 20 members.

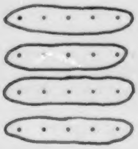
---

*Solutions to problems will be found on page 147.

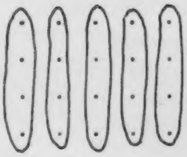
$$\begin{array}{llllll} B \times A = \{ & (\alpha,a) & (\alpha,b), & (\alpha,c), & (\alpha,d), \\ & (\beta,a), & (\beta,b), & (\beta,c), & (\beta,d), \\ & (\gamma,a), & (\gamma,b), & (\gamma,c), & (\gamma,d), \\ & (\delta,a), & (\delta,b), & (\delta,c), & (\delta,d), \\ & (\epsilon,a), & (\epsilon,b), & (\epsilon,c), & (\epsilon,d)\}. \end{array}$$

Therefore, even though $A \times B \neq B \times A$, both product sets are equivalent; that is, they have the same number property.

Notice from the above displays that an array of 5 disjoint sets each having 4 members, and an array of 4 disjoint sets, each having 5 members, have the same number property.



4 sets, 5 members
in each set



5 sets, 4 members
in each set

Since multiplication refers only to the number properties of sets involved in the cartesian product, the fact that the cartesian product is not commutative has no bearing on the commutativity under multiplication. It is still true that the set of whole numbers is commutative under multiplication; that is

FOR ANY WHOLE NUMBERS A AND B, $A \times B = B \times A$.

In the example that we have used, $4 \times 5 = 5 \times 4$. A 4 by 5 array has the same number of members as a 5 by 4 array.

The array as a union of 4 disjoint sets, each having 5 members also shows that 4×5 can be computed by the successive addition.

$$\overbrace{5 + 5 + 5 + 5}^{4 \text{ addends}} ,$$

that is, 5 is used as an addend 4 times. (This is sometimes referred to as the repeated addition description of multiplication.)

Although multiplication of whole numbers may be described in terms of repeated addition, it must be remembered that multiplication is defined as an operation on two numbers and is independent of addition. The operation showing the association of a third number with a given pair may be indicated, for example, by the usual method: $4 \times 5 = 20$ or simply $(4,5) \longrightarrow 20$. "$(4,5) \longrightarrow 20$" may be read: "to 4 and 5 is assigned the number, 20". Likewise, addition may be so described; thus $(4, 5) \longrightarrow 9$ may refer to an operation of addition.

PROBLEMS

3. Draw two arrays to illustrate that $3 \times 4 = 4 \times 3$.

4. Is it possible to draw an array to illustrate 3×0? Why or why not?

5. For each operation given below, state which arithmetic operation it refers to.

   a. $(2,5) \longrightarrow 10$
   b. $(3,5) \longrightarrow 8$
   c. $(5,0) \longrightarrow 5$
   d. $(1,1) \longrightarrow 1$
   e. $(1,1) \longrightarrow 2$
   f. $(2,2) \longrightarrow 4$

6. In adding, there is a particular number a such that $a + a = a$; find this number.

7. In multiplication, is there a number such that $a \times a = a$? Is there more than one number a such that $a \times a = a$?

We have defined multiplication as a binary operation, that is, it is an operation on two numbers at a time. To find the product of three numbers, for example 3, 4, and 5, we may multiply 3 and 4 and get the product 12. We know that this product is a whole number because the set of whole numbers is closed under the operation of addition. We then multiply 12 and 5 and get the product 60.

We write this

$$\begin{aligned}(3 \times 4) \times 5 &= 12 \times 5\\ &= 60.\end{aligned}$$

We might have multiplied 4 and 5 getting the product 20, and then multiplied 3 and 20 getting the product 60. We write this

$$\begin{aligned}3 \times (4 \times 5) &= 3 \times 20\\ &= 60.\end{aligned}$$

In either case, the product is the same; that is

$$(3 \times 4) \times 5 = 3 \times (4 \times 5).$$

Observation of several examples and our intuition convince us that the order in which we associate the factors in multiplication does not affect the product.

This is true in general.

FOR ANY WHOLE NUMBERS a, b, and c

$$(a \times b) \times c = a \times (b \times c)$$

This is called the associative property of whole numbers under multiplication.

For the example that we used above,

$$(3 \times 4) \times 5 = 12 \times 5 = 60$$

and

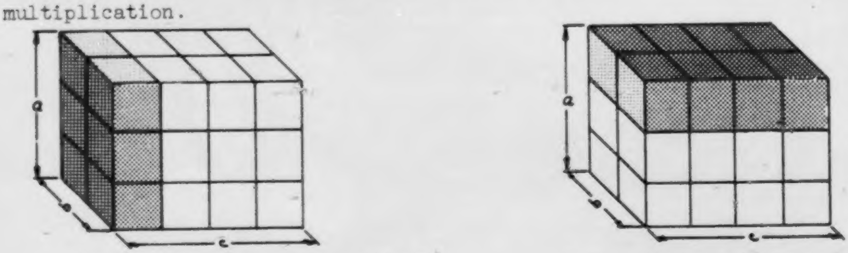
$$3 \times (4 \times 5) = 3 \times 20 = 60.$$

Alternately, this may be written as follows:

$$\begin{array}{ccc} (3 \times 4) \times 5 & & 3 \times (4 \times 5) \\ \parallel & & \parallel \\ 12 \times 5 & & 3 \times 20 \\ \parallel & & \parallel \\ 60 & = & 60 \end{array}$$

Showing again that $(3 \times 4) \times 5 = 3 \times (4 \times 5)$ by virtue of the statement, $60 = 60$; that is to say, both expressions name the same number.

The physical model of a box made up of cubical blocks with dimensions a by b by c, may be used to illustrate the associativity of multiplication.



$a \times b$ blocks in each vertical slice; c vertical slices.

$b \times c$ blocks in each horizontal slice; a horizontal slices.

Model illustrating the associative property of multiplication.

The number of blocks in such a box is $(a \times b) \times c$ and is also $a \times (b \times c)$ indicating that it is true that $(a \times b) \times c = a \times (b \times c)$.

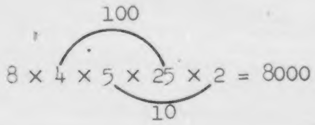
PROBLEMS

8. Show that $2 \times 3 \times 4 = 8 \times 3$ involves both the commutative and the associative properties of multiplication.

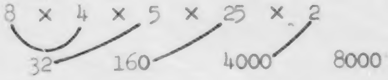
9. What property or properties are involved in each of the following?

   a. $2 \times 3 \times 4 = 2 \times 12$
   b. $2 \times 3 \times 4 = 3 \times 8$
   c. $2 \times 3 \times 4 = 6 \times 4$
   d. $2 \times 3 \times 4 = 2 \times 4 \times 3$
   e. $2 \times 3 \times 4 = 3 \times 2 \times 4$
   f. $4 \times 3 \times 2 = 4 \times 3 \times 2$

Just as we could "pick and choose" pairs of addends in a sum, the commutative and associative properties under multiplication allow us to "pick and choose" pairs of factors in a product. For example,



Natural combinations yielding tens, hundreds, and so on might make for ease in computations. To be sure, for the same product, one can proceed to compute laboriously as follows:



PROBLEM

10. Show by grouping with parentheses how $a \times b \times c \times d$ may be regarded as a product involving 3 factors instead of 4 for each of the following:

a. $2 \times 3 \times 4 \times 5 = 2 \times 3 \times 20$
b. $2 \times 3 \times 4 \times 5 = 6 \times 4 \times 5$
c. $2 \times 3 \times 4 \times 5 = 2 \times 12 \times 5$

The number 1 occupies, with respect to multiplication, the same position that 0 occupies with respect to addition. Notice that,

$$\begin{aligned} 1 \times 3 = 3 \times 1 = 3, \\ 1 \times 5 = 5 \times 1 = 5, \\ 1 \times 6 = 6 \times 1 = 6, \\ 1 \times 8 = 8 \times 1 = 8. \end{aligned}$$

It is true that $1 \times a = a$ for all numbers a because a 1 by a array consists of only one row having a members, and therefore the entire array contains exactly a members.

$$1\{\overbrace{.\ .\ .\ .\ .}^{5}\} \qquad 1\{\overbrace{.\ .\ .\ .\ .\ .}^{6}\} \qquad 1\{\overbrace{.\ .\ .\ .\ .\ .\ .\ .}^{8}\}$$

$$1 \times 5 = 5 \qquad 1 \times 6 = 6 \qquad 1 \times 8 = 8$$

Since $1 \times a = a$, the number 1 is called the identity element for multiplication. The property is referred to as the property of 1 under multiplication:

FOR EVERY WHOLE NUMBER a, $1 \times a = a$.

Because of the commutative property under multiplication, we also have $a \times 1 = a$.

While 0 does not act as the identity in multiplication, it does have a special role. The number of members in a 0 by 3 array (that is, an array with 0 rows, each with 3 members) is 0 because the set of members of this array is empty. In general, if a is a whole number, the number of members in a 0 by a array is 0; thus,

FOR EVERY WHOLE NUMBER a, $0 \times a = 0$.

It is also true that $a \times 0 = 0$.

The characteristics of 0 in multiplication of "annihilating" (so to speak) all numbers except 0 in the product has an important consequence. If any factor is 0, the product is 0.

What has been done so far shows that multiplication, as well as addition, is an operation on the whole numbers which has the properties of closure, commutativity and associativity. There is a special number 1 that is an identity for multiplication just as 0 is an identity for addition. Moreover, 0 plays a special role in multiplication for which there is no corresponding property in addition.

There is another important property that links the operations of addition and multiplication. This property which we shall now study is the basis, for example, for the following statement:

$$4 \times (7 + 2) = (4 \times 7) + (4 \times 2).$$

This example may be verified by noting that both $4 \times (7 + 2)$ and $(4 \times 7) + (4 \times 2)$ give the same result:

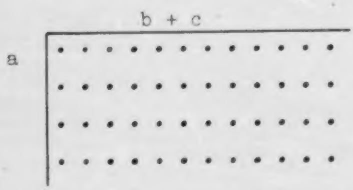
$$4 \times (7 + 2) = 4 \times 9 = 36, \text{ and}$$

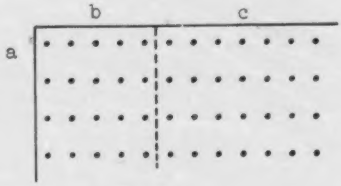
$$(4 \times 7) + (4 \times 2) = 28 + 8 = 36.$$

The property is called the *distributive property of multiplication over addition*. The distributive property states that if a, b and c are any whole numbers, then

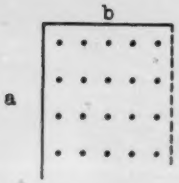
$$a \times (b + c) = (a \times b) + (a \times c) .$$

The distributive property may be illustrated by considering an a by (b + c) array.

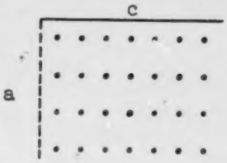




It is true that this array is formed from an a by b array and an a by c array.



An a by b array



An a by c array

Consequently, the number $a \times (b + c)$ of members in the large array is the sum of $(a \times b)$ and $(a \times c)$, the numbers of members of the subsets. That is, $a \times (b + c) = (a \times b) + (a \times c)$.

Since multiplication is commutative, both the "left hand" and the "right hand" distributive properties hold, that is,

Left hand: $a \times (b + c) = (a \times b) + (a \times c)$, and
Right hand: $(b + c) \times a = (b \times a) + (c \times a)$.

For example, by these distributive properties,

Left hand: $3 \times (5 + 8) = (3 \times 5) + (3 \times 8)$, and
Right hand: $(4 + 7) \times 2 = (4 \times 2) + (7 \times 2)$.

Recalling that when we say $A = B$ we mean A and B both name the same thing, then if $A = B$, it really makes no difference whether we write $A = B$ or $B = A$. With this in mind, since the left hand distributive property says that $a \times (b + c)$ and $(a \times b) + (a \times c)$ both name the same number, the statement

$$a \times (b + c) = (a \times b) + (a \times c)$$

can equally well be written as

$$(a \times b) + (a \times c) = a \times (b + c).$$

For example,

$$(3 \times 5) + (3 \times 8) = 3 \times (5 + 8).$$

Similarly, the right hand distributive property may be expressed as either

$$(b + c) \times a = (b \times a) + (c \times a)$$

or

$$(b \times a) + (c \times a) = (b + c) \times a .$$

For example,

$$(4 \times 2) + (7 \times 2) = (4 + 7) \times 2.$$

The distributive property is very important as it is the basis for computing the product of two numbers.

$$\begin{aligned} \text{Left hand:}\quad (5 \times 4) + (5 \times 6) &= 5 \times (4 + 6) \\ &= 5 \times 10 = 50;\ \text{also} \\ \text{Right hand:}\quad (7 \times 9) + (3 \times 9) &= (7 + 3) \times 9 \\ &= 10 \times 9 = 90. \end{aligned}$$

The convenience may be further illustrated by the following examples:

$$\begin{aligned} &(9 \times 17) + (9 \times 83) = 9 \times (17 + 83) = 9 \times 100 = 900; \\ &(24 \times 17) + (26 \times 17) = (24 + 26) \times 17 = 50 \times 17 = 850; \\ &(854 \times 673) + (146 \times 673) = (854 + 146) \times 673 = 1000 \times 673 = 673{,}000; \\ &(84 \times 367) + (84 \times 633) = 84 \times 1000 = 84{,}000. \end{aligned}$$

PROBLEMS

11. Use the distributive property to compute each of the following:

    a. $(57 \times 7) + (57 \times 93)$

    b. $(57 \times 8) + (57 \times 93)$ [Hint: $8 = 7 + 1$]

12. Show that $(57 \times 5) + (57 \times 5) = 57 \times 10$ by the distributive property.

One might question whether addition distributes over multiplication. That is, is it always the case that

$$a + (b \times c) = (a + b) \times (a + c)?$$

This would be false if any set of numbers a, b and c can be found that would disprove the statement. For example, $a = 1$, $b = 3$, and $c = 2$ may be tried. For these values,

$$\begin{aligned} &a + (b \times c) = 1 + (3 \times 2) = 1 + 6 = 7,\ \text{but} \\ &(a + b) \times (a + c) = (1 + 3) \times (1 + 2) = 4 \times 3 = 12\,. \end{aligned}$$

So it cannot be stated that $a + (b \times c)$ is always equal to $(a \times b) + (a \times c)$.

SUMMARY OF PROPERTIES

The properties of multiplication developed so far for whole numbers may be summarized as follows, where a, b, and c are whole numbers.

1. Whole numbers are CLOSED under multiplication:

$$a \times b \text{ is a whole number.}$$

2. Multiplication is a COMMUTATIVE operation:

$$a \times b = b \times a.$$

3. Multiplication is an ASSOCIATIVE operation:

$$(a \times b) \times c = a \times (b \times c).$$

4. There is an IDENTITY element 1 for multiplication:

$$a \times 1 = a.$$

5. Multiplication is DISTRIBUTIVE over addition:

$$a \times (b + c) = (a \times b) + (a \times c).$$

6. Zero has a special multiplication property:

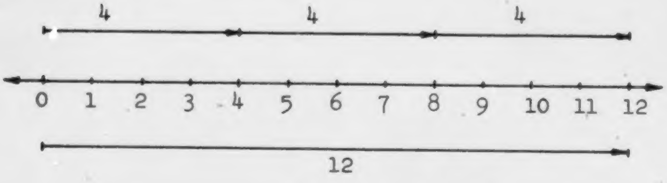
$$0 \times a = 0.$$

MULTIPLICATION USING THE NUMBER LINE

Through the interpretation of multiplication as repeated addition, multiplication may be illustrated on the number line. For example, 3×4 means 3 addends, each addend being 4. That is,

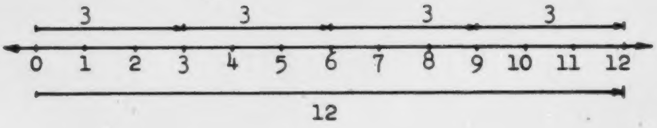
$$3 \times 4 = 4 + 4 + 4.$$

Therefore, this may be represented by 3 successive arrows as shown below:



$3 \times 4 = 12$

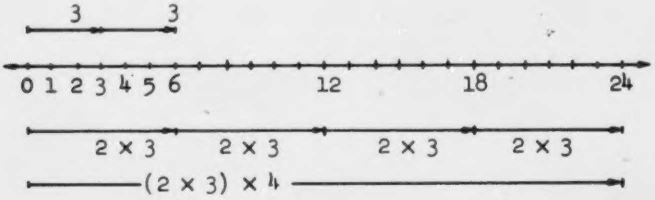
On the other hand, 4 × 3 means 4 addends of 3. The representation on the number line is as follows:



4 × 3 = 12

As we can see, the two representations above are different; however, both of these yield the same result. By combining these two in a single diagram, we illustrate the commutative property under multiplication.

When more than two factors are involved, this too may be illustrated. For example, to show (2 × 3) × 4, we have the following.



(2 × 3) × 4 = 24

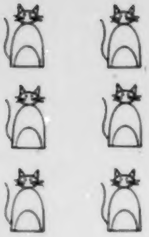
Likewise, 2 × (3 × 4) may be shown by obtaining two (3 × 4) "arrows" and abutting them. By combining the diagrams for (2 × 3) × 4 and 2 × (3 × 4), associativity may be illustrated.

PROBLEM

13. Represent multiplication on the number line for 2 × (3 × 4).

APPLICATIONS TO TEACHING

We introduce the array as a means of providing readiness for the concept of multiplication. The rectangular arrangements of flannel board objects, blocks on the floor, panes in the window, eggs in a carton may all be described as arrays. If we have an array such as



lead the children to recognize that this may be thought of as three sets of two cats in each set or as two sets of three cats in each set.

Commutativity under multiplication may be conveyed by arranging chairs facing the board, for example, in an array of 10 rows, 2 to each row. When the chairs are turned 90° from the original direction, there will be 2 rows, 10 to each row. In each case $(10 \times 2$ or $2 \times 10)$, the number of children is 20.

The associative and distributive properties are not presented until the second grade. To illustrate the distributive property 4 sacks, each containing, say, 5 red blocks and 3 yellow blocks may be used. Thus, in the 4 sacks, there are 20 red blocks and 12 yellow blocks, or, 32 blocks.

$$4 \times (5 + 3) = (4 \times 5) + (4 \times 3).$$

QUESTION

"Is there any practical situation that requires students to know what the distributive property is all about?"

The answer to this question depends on what is meant by "practical". As indicated in the text, the question for example, of the auditorium seating may be a very practical situation to the children, or the fact that there are occasions when one can make computation easier may be very practical to some. Learning to recognize that

$$a \times (b + c) = (a \times b) + (a \times c)$$

and

$$(a \times b) + (a \times c) = a \times (b + c)$$

say exactly the same thing is quite important. Later, this is applied to factoring many expressions as a step in solving equations. That is a very practical situation for some students.

Aside from this we make use of this property whenever we multiply by numbers named by two or more digits. The fact that we multiply each of the digits, 7 and 2, individually by 4 in the problem 72 × 4 is based upon this property:

$$\begin{array}{r} 72 \\ \times\ 4 \\ \hline 288 \end{array}$$

This is because $72 \times 4 = (70 + 2) \times 4$ since 72 and $(70 + 2)$ are names for the same number. Thus

$$\begin{aligned} 72 \times 4 &= (70 + 2) \times 4 \\ &= (70 \times 4) + (2 \times 4) \\ &= 280 + 8. \end{aligned}$$

This same property may be applied to smaller numbers. For example, 5×8 may be rewritten $5 \times (3 + 5)$ since 8 and $(3 + 5)$ name the same number. By the distributive property,

$$5 \times (3 + 5) = (5 \times 3) + (5 \times 5).$$

Thus, a "large" factor may be broken down as the sum of two or more smaller addends; in this case, 8 is thought of as $3 + 5$. Although there are activities in Book 1 leading to this property, the topic is not openly treated until the end of Grade 2.

VOCABULARY

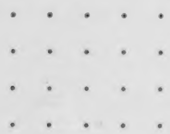
Array*
Associative Property
    of Multiplication *
Closure Property of Whole
    Numbers under Multiplication*
Distributive Property of
    Multiplication over Addition*
Factor*
Identity Element*
Multiplication*
Product*
Product Set*
Property of One under Multiplication*

EXERCISES - CHAPTER 8

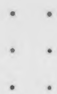
1. Show by trying to indicate the steps in repeated addition how the commutative property of multiplication would simplify the calculation of 1000 × 3.

2. What mathematical sentence is suggested by each of the arrays below?

a.



b.

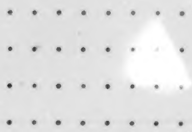


3. Mr. Rhodes is buying a two-tone car. The company offers tops in 5 colors and bodies in 3 colors. Draw an array that shows the various possible results, assuming that none of the body colors are the same as any of the top colors.

4. Mr. Rhodes is buying a two-tone car. Colors available for the top are: red, orange, yellow, green and blue. Colors available for the body are: red, yellow and blue. Draw an array to show the various possible results. If Mr. Rhodes insists that the car must be two-toned, how many choices does he have?

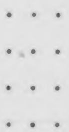
5. An ensemble of sweater and skirt is offered with the sweater available in five different colors and the skirt in 4 colors. The skirt also comes in either straight or flare style for each of the 4 colors. How many different ensembles are possible?

6. Here is an array separated into two smaller arrays.



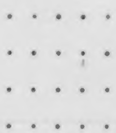
$(n = 4 \times 8)$

Array A



$(p = 4 \times 3)$

Array B



$(q = 4 \times 5)$

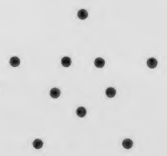
Array C

a. How many dots are in Array A? Array B? Array C?

b. Does $n = p + q$?

c. Does $4 \times 8 = (4 \times 3) + (4 \times 5)$?

7. A familiar puzzle problem calls for planting 10 trees in an orchard so there are 5 rows with 4 trees in each row. The solution is in the form of the star shown in the figure to the right. Why doesn't this star illustrate the product of 5 and 4?

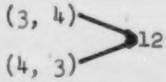


8. The middle section of an auditorium seats 23 to a row, and each side section seats 11 to a row. What is the capacity of this auditorium if there are 20 such rows?

9. Use the commutative and associative properties to get the answer quickly by "picking and choosing" appropriate combinations:

   a. 5 × 4 × 3 × 2 × 1
   b. 125 × 7 × 3 × 8
   c. 250 × 14 × 4 × 2

10. What does the following operation indicate for 3 × 4?



11. Make each of the following a true statement illustrating the distributive property.

   a. 3 × (4 + ___) = (3 × 4) + (3 × 3)
   b. 2 × (___ + 5) = (2 × 4) + (___ × 5)
   c. 13 × (6 + 4) = (13 × ___) + (13 × ___)
   d. (2 × 7) + (3 × ___) = (___ + ___) × 7

SOLUTIONS FOR PROBLEMS

1. a. o o o o o
o o o o o
o o o o o

b. o o
o o
o o
o o
o o
o o

c. o o o o o o
o o o o o o

d. o o o
o o o
o o o
o o o
o o o
o o o
o o o

e. o o o o o
o o o o o
o o o o o
o o o o o
o o o o o
o o o o o
o o o o o
o o o o o

f. o o o o
o o o o
o o o o
o o o o
o o o o
o o o o
o o o o
o o o o
o o o o

2. $A \times B = \{(a,0),(a,\triangle),(a,\square), (a,\boxtimes), (b,0),(b,\triangle),(b,\square), (b,\boxtimes)\}$

$B \times A = \{(0,a), (0,b), (\triangle,a), (\triangle, b), (\boxtimes,a), (\boxtimes,b)(\square,a)(\square,b)\}$

$N(A \times B) = 8$, $N(B \times A) = 8$, $A \times B \neq B \times A$, $N(A \times B) = N(B \times A)$.

3. o o o o   o o o
o o o o   o o o
o o o o   o o o
3 × 4   o o o
4 × 3

4. No, 3 × 0 is the number property of the empty set.

5. a. multiplication
   b. addition
   c. addition
   d. multiplication
   e. addition
   f. addition or multiplication

6. $a = 0$

7. Yes; either $a = 0$ or $a = 1$

8. $2 \times 3 \times 4 = 2 \times (3 \times 4)$ associative property
   $= 2 \times (4 \times 3)$ commutative property
   $= (2 \times 4) \times 3$ associative property
   $= 8 \times 3$ renaming

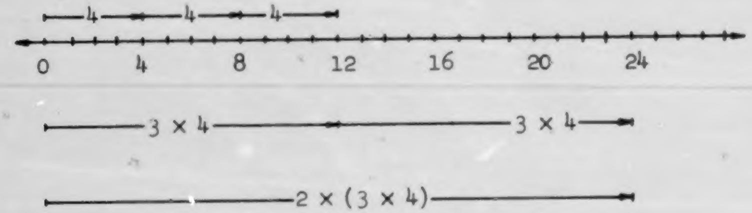
9. a. $2 \times 3 \times 4 = 2 \times (3 \times 4) = 2 \times 12$ associative
   b. $2 \times 3 \times 4 = (2 \times 3) \times 4$ associative
   $= (3 \times 2) \times 4$ commutative
   $= 3 \times (2 \times 4)$ associative
   $= 3 \times 8$
   c. $2 \times 3 \times 4 = (2 \times 3) \times 4 = 6 \times 4$ associative
   d. $2 \times 3 \times 4 = 2 \times 4 \times 3$ commutative
   e. $2 \times 3 \times 4 = 3 \times 2 \times 4$ commutative
   f. $4 \times 3 \times 2 = 4 \times 3 \times 2$ none involved

10. a. $2 \times 3 \times 4 \times 5 = 2 \times 3 \times (4 \times 5) = 2 \times 3 \times 20$
    b. $2 \times 3 \times 4 \times 5 = (2 \times 3) \times 4 \times 5 = 6 \times 4 \times 5$
    c. $2 \times 3 \times 4 \times 5 = 2 \times (3 \times 4) \times 5 = 2 \times 12 \times 5$

11. a. $(57 \times 7) + (57 \times 93) = 57 \times (7 + 93) = 57 \times 100 = 5700$
    b. $(57 \times 8) + (57 \times 93) = (57 \times (1 + 7)) \times (57 + 93)$
    $= (57 \times 1) + (57 \times 7) + (57 \times 93) = (57 \times 1) + (57 \times (7 + 93))$
    $= (57 \times 1) + (57 \times 100) = 57 + 5700 = 5757$

12. $(57 \times 5) + (57 \times 5) = 57 \times (5 + 5) = 57 \times 10 = 570$

13.



Chapter 9

SUBTRACTION

THE REMAINING SET

If $A = \{\text{Cornelia, Sally, Jimmy, Emily, Elsie, Edward, Douglas}\}$ and if $B = \{\text{Cornelia, Sally, Emily, Elsie}\}$, then B is a subset of A. When B is specified as a subset of A, another subset of A is simultaneously specified; namely, by all the elements of A that are *not* elements of B. In this way, an operation is defined, producing from A and B, a set called the *complement of B relative to A*, or more simply, the *remaining set*. Thus, if $C = \{\text{Jimmy, Edward, Douglas}\}$, and A and B are as above, then C is the remaining set.

Together, the union of B and C is A, so the two subsets "complete" the given set. Since C is composed of elements that are not elements of B, it is clear that the intersection of B and C is the empty set. In fact, these last two statements can be used as the basis for defining the relative complement, or remaining set. We denote the operation by the symbol "-". For example, if $A = \{0, \triangle, \square, \bigstar, ☽\}$ and $B = \{0, \square\}$, then $A - B = \{\triangle, \bigstar, ☽\}$. This is read "The relative complement of B to A is the set $\{\triangle, \bigstar, ☽\}$". Of course, the goal is to connect this operation with subtraction, and this goal is immediately achieved by looking at the appropriate number properties. Note that in this example, the number property of A is 5, the number property of B is 2, and the number property of $A - B$ is 3. In general, it is true that

$$N(A - B) = N(A) - N(B).$$

Since the definition of $A - B$ requires B to be a subset of A, there are evidently restrictions on B. B can be the empty set; B can be identical to A; these two sets, A and the empty set, establish the limits on B. Consequently, if $N(A) = a$ and $N(B) = b$, we have the restrictions $b \geq 0$ and $b \leq a$. (The symbol "$\geq$" combines ">" and "=" to indicate "is greater than *or* equal to"; similarly "$\leq$" is read "is less than *or* equal to".) The restrictions can be incorporated into the one statement, $0 \leq b \leq a$; that is, the number of elements in B can range from 0 to the number of elements in A. These limitations for subtraction are eventually relaxed when the set of numbers that we have to work with is extended to include more than just the whole numbers. The pattern of development proceeds thus: from observations on complementation, the characteristics of subtraction are examined; from examination of the characteristics, the operation is extended. As a result, numbers other than whole numbers may be introduced. For example,

$$\text{if } A = \{a, b, c, d, e\} \text{ and}$$
$$B = \{a, b, c\}, \text{ then } A - B = \{d, e\}.$$

From this, we get the difference

$$N(A) - N(B) = N(A - B); \text{ that is}$$
$$5 - 3 = 2.$$

The statement, $5 - 3 = 2$, may in turn trigger the question whether subtraction may be defined for any two whole numbers. For example, is $5 - 8$ defined? If we limit ourselves to the set of whole numbers, the answer is "no". But by reassessing the behavior of subtraction, it is possible to introduce new members to the number system so that subtraction is always defined in the system.

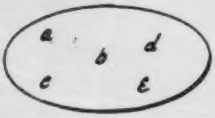
The example, $5 - 8$, brings out two important features of the subtraction operation. Since no whole number is the result of $5 - 8$, the set of whole numbers is not closed under subtraction. Contrasted with $8 - 5$, which does yield a whole number for an answer, we see that in general, if a and b are whole numbers, it is not true that $a - b$ is the same as $b - a$. Thus, subtraction is neither closed nor commutative. These are negative results; they tell us some of the properties that subtraction does not have. Nevertheless, these are important results.

SUBTRACTION AS INVERSE

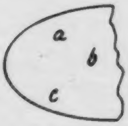
Subtraction is not restricted to only negative results, however; nor is the operation of getting remaining sets so restricted. A noteworthy result may be stated thus:

$$(A - B) \cup B = A .$$

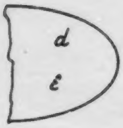
In words: If we form the remaining set $A - B$, and then form the union of it with B, we have the original set, A. Diagrammatically, the situation may be illustrated as follows:



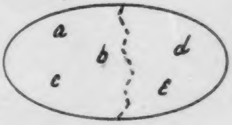
A



B



A - B



$(A - B) \cup B$

Similarly, if we start out with a set, X, and join a disjoint set Y to it, we get $X \cup Y$. Now if we take the complement of Y relative to $X \cup Y$, then we have $(X \cup Y) - Y$, which turns out to be X, the original set. That is,

$$(X \cup Y) - Y = X.$$

Because of these two situations, we say that union and complementation are inverse operations. In effect, one operation "undoes" what is done by the other. Corresponding to these properties under set operations, we have similar properties under addition and subtraction:

IF a AND b ARE WHOLE NUMBERS, AND $b \leq a$, THEN $(a - b) + b = a$; AND, IF a AND b ARE ANY WHOLE NUMBERS $(a + b) - b = a$.

Therefore, subtraction and addition are inverse operations whenever the two operations are possible or defined.

DEFINITIONS OF SUBTRACTION

We have defined the difference as the number property of the remaining set. This gives us a means of finding $a - b$ if a is a number and if b is a number less than or equal to a. We first choose a set, A, such that $N(A) = a$; next we pick a set, B, which is a subset of A and such that $N(B) = b$, $b \leq a$. These two sets determine the remaining set, A - B. The number, $a - b$, is the number of elements in A - B:

$$a - b = N(A - B).$$

For example, if $a = 5$ and $b = 2$, we can choose A to be the set

$$A = \{0, \triangle, \square, \bigstar, \mathcal{E}\}.$$

Next we can choose B to be the subset

$$B = \{\triangle, \bigstar\}.$$

Then

$$A - B = \{0, \square, ℰ\}.$$

Now our definition tells us that

$$5 - 2 = N(A - B) = 3.$$

Note that if we made a different choice for B, for example

$$B = \{\square, ℰ\},$$

the result would be the same as far as the number property is concerned. Also, if we had chosen a different set, A, for example $A = \{V, W, X, Y, Z\}$, and any two member subset of this set as B, the result would still be the same.

PROBLEM *

1. Use the above definition of subtraction to compute in detail $7 - 3$.

There is a second approach to subtraction which does not use the idea of the remaining set, but uses the ideas of union of disjoint sets and of one-to-one correspondence. If a is a number and if b is a number with $b \leq a$, we start by choosing a set A with $N(A) = a$ and a set B disjoint from A with $N(B) = b$.

Next we choose a set C, disjoint from both A and B in such a way that A and $(B \cup C)$ are in one-to-one correspondence. That is, there is a pairing of the elements of A with the elements of $B \cup C$. Then the second definition of subtraction is:

$$a - b = N(C).$$

In other words, having chosen appropriate disjoint sets A and B we look for a third set C with just the right number of members so that the union of this set and the set B will exactly match with the set A. The number of members in such a set C tells us "how many more members" A has than B.

As an example of this definition of subtraction let us again use $a = 5$ and $b = 2$. A can be the same set $\{0, \triangle, \square, \star, ℰ\}$ as was used before, but B must now be a disjoint set with 2 members. Let $B = \{X, Y\}$. An attempt to get a one-to-one correspondence between the elements of B and the elements of A may result in the following,

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*Solutions for problems in this chapter are on page 162.

$$\begin{array}{rcccccl} B = \{ & X, & Y\} & & & & \\ & \downarrow & \downarrow & & & & \\ A = \{ & O, & \triangle, & \square, & \star, & \mathcal{E} & \}, \end{array}$$

leaving some elements of A unpaired. We look for a set, C (disjoint from B) so that $B \cup C$ will match A. Thus, if $C = \{\alpha, \beta, \delta\}$, then the elements of $B \cup C$ can be put into one-to-one correspondence with those of A.

$$\begin{array}{rcccccl} B \cup C = \{ & X, & Y, & \alpha, & \beta, & \delta & \} \\ & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \\ A = \{ & O, & \triangle, & \square, & \star, & \mathcal{E} & \}. \end{array}$$

Now by the second definition of subtraction, the result of $5 - 2$ is the number property of C. Therefore, $5 - 2 = N(C) = 3$. The most important thing to say about this definition of subtraction is that it always gives exactly the same result as the first definition.

PROBLEM

2. Use the second definition given above of subtraction to compute in detail $7 - 3$.

Now the question naturally arises as to why we should bother with two different definitions if they both give the same result. Why not use just one of them?

The reason is that there are two quite different kinds of problems that we commonly meet and it is important to know that the same mathematical operation can be used to solve both kinds of problems.

The first kind is the "take away" type:

> "Fred has 5 dollars and loses two of them. How many dollars does he have left?"

The second kind is the "how many more" type:

> "Fred has 5 dollars. Bill has 2 dollars. How many more dollars does Bill need in order to have as many as Fred?"

The first definition of subtraction fits very well with the "take away" type of problem, and the second fits very well with the "how many more" type. But in each case the problem is solved by means of the subtraction: $5 - 2 = 3$.

The statement that we have on page 148, relating addition to to subtraction, namely

$$(a - b) + b = a,$$

gives us yet another insight into the concept of subtraction. If $a - b$ is some number c, then we have

$$c + b = a.$$

In other words, $a - b$ is that number c such that $a = c + b$. This is why we can say that

$$a - b = c \text{ IF AND ONLY IF } a = c + b;$$

these two statements mean exactly the same thing.

From this point of view, subtraction is defined as the operation of finding the unknown addend, c, in the addition problem

$$a = c + b$$

since this is the same number as $a - b$. For example, we can state that $5 - 2$ is $\underline{3}$ because $5 = \underline{3} + 2$.

Also, since we know that both

$$5 = 3 + 2 \text{ and } 5 = 2 + 3$$

it is true that

$$5 - 2 = 3 \text{ and } 5 - 3 = 2.$$

Except as noted in Problem 4 below, an addition fact gives us two related subtraction facts.

PROBLEMS

3. The two statements $a - b = c$ and $a = c + b$ mean the same thing. Working with whole numbers 6, 4, and 2 show the related addition and subtraction facts.
4. When does an addition fact not give two subtraction facts.

There are two reasons why it is important for teachers to understand this way of thinking about subtraction, as well as the other two. The first is that this is the way that children usually think when they are developing their skills in computation. The second is that as children move through school, and study other kinds of numbers, such as rational decimals, negative numbers, etc., they will meet this idea of defining subtraction in terms of addition again and again.

It is important to realize that all three definitions of subtraction are equivalent and yield the same properties.

PROPERTIES UNDER SUBTRACTION

We have noted the property of subtraction that points to its role as an inverse of addition. Two properties of the whole numbers under this operation that we want to highlight now involve the empty set. Recall that with the union, we have

$$A \cup \{\ \} = A.$$

The corresponding statement for numbers is for any whole number a,

$$a + 0 = a.$$

By the above, we observe that

$$a + 0 = a \quad \text{and} \quad a = a - 0$$

say the same thing. Since $a + 0 = 0 + a$, we also have $0 + a = a$, which is the same as $0 = a - a$. Hence, in addition to the inverse properties,

FOR ANY WHOLE NUMBERS a and b, WITH $a \geq b$, $(a - b) + b = a$,

FOR ANY WHOLE NUMBERS a and b, $(a + b) - b = a$.

we have the following two properties of zero under subtraction:

FOR ANY WHOLE NUMBER a, $a - 0 = a$;

FOR ANY WHOLE NUMBER a, $a - a = 0$.

PROBLEMS

5. By a definition of subtraction, we see that $a - b = c$ if and only if $a = c + b$, and that $(a - b) + b = a$. Which properties are exemplified by the following?
   a. $(202 - 200) + 200 = 202$
   b. $(y - x) + x = y$
   c. $[(30 - 15) - 5] + 5 = 15$
   d. $5 + 0 = 5$
   e. $5 - 0 = 5$

6. Does the sentence $(5 - 7) + 7 = 5$ make sense for whole numbers?

7. Show by the use of the properties of addition and subtraction that the following sentence is true:

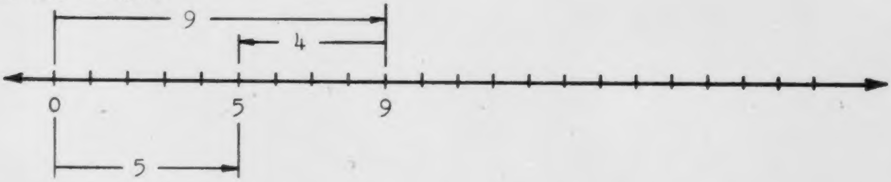
$$\text{If } b \geq a, \quad a + (b - a) = b.$$

Check that it is true by using several pairs of numbers.

SUBTRACTION USING THE NUMBER LINE

If we consider subtraction with respect to the representation of numbers using the number line, we can illustrate many of its important processes and properties.

What is the answer to $9 - 4$? We start on the number line at 9 and take away or move to the left 4 units thus arriving at 5, which is our answer.



$$9 - 4 = 5$$

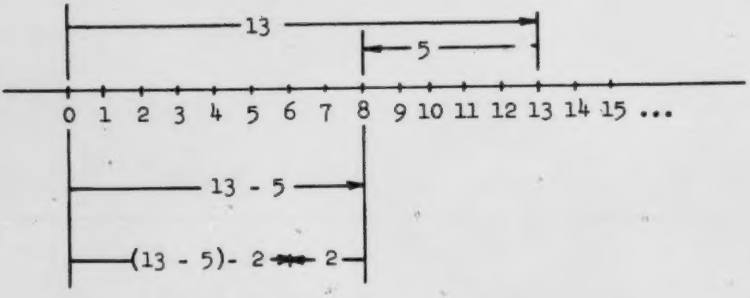
In Chapter 7 we illustrated the use of the number line to show the associative property of addition. Subtraction does not have the associative property for

$$(13 - 5) - 2 = 8 - 2 = 6$$

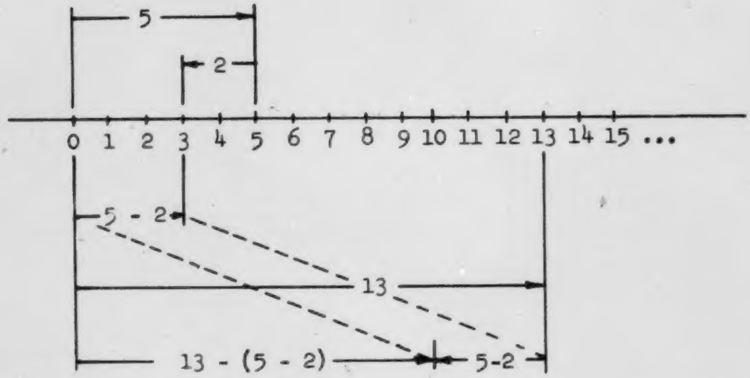
while

$$13 - (5 - 2) = 13 - 3 = 10.$$

These examples are illustrated on number lines below. The first figure shows that $13 - 5 = 8$, and this result is used to get 6 from $8 - 2$. The second shows that $5 - 2 = 3$, and this result is used to get 10 from $13 - 3$.


13
5
0 1 2 3 4 5 6 7 8 9 10 11 12 13 14 15 ...
13 - 5
(13 - 5)- 2
2

$$(13 - 5) - 2 = 8 - 2 = 6$$


5
2
0 1 2 3 4 5 6 7 8 9 10 11 12 13 14 15 ...
5 - 2
13
13 - (5 - 2)
5-2

$$13 - (5 - 2) = 13 - 3 = 10$$

Hence, it is not true that $(13 - 5) - 2$ names the same number as $13 - (5 - 2)$ and we express this by the number sentence.

$$(13 - 5) - 2 \neq 13 - (5 - 2),$$

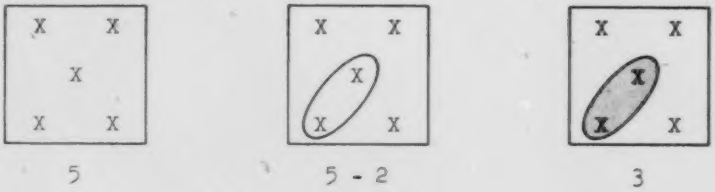
where the symbol "$\neq$" means "is not equal to".

APPLICATIONS TO TEACHING

Some children find it difficult to visualize set removal. For them partitioning and ringing a subset is not enough; they cannot seem to appreciate that the objects have been removed since the objects are still much in evidence. Covering up the objects to be removed or crossing them out with an ✕ may help communicate removal. Similarly, using a cup to cover up a subset of beans, for example, has been found to be effective in teaching set removal.

On the other hand, removal may have been so convincing that it causes difficulty with writing the number sentence associated with the removal. For example, in trying to connect the expression 5 - 2 with 3, only the numbers for the original set and the remaining set may be recorded; the other subset has been removed, so the child cannot understand why its number must be recorded. In that case, intermediate stages in the removing process may be suggested. This may be in the form of a class activity, for example, with a set of beans. The number of the set may first be recorded; a subset may next be separated, counted, and the number recorded. Removal may be accomplished by covering the set removed (as with a cup) and finally, the number in the remaining set identified and recorded.

Intermediate stages for the recording of numbers in the ringing of set members may also be provided. For example, the following suggests various possible stages for 5 - 2 = 3.



The concept of inverse may prove difficult. For this, a variety of examples may be required showing situations which have inverses such as falling asleep and waking up, say, or putting on a coat and taking it off. However, sometimes it is not the lack of understanding of the concept that is causing difficulty; it may be trying to verbalize the "doing and undoing" that the children find difficult.

QUESTION

"What is meant by the statement a - b = c if and only if a = c + b?"

The statement is equivalent to two separate statements, "a - b = c if a = c + b" and "a = c + b if a - b = c". Its application here for example, may be in arriving at the solution to 5 - 3 by finding the "missing addend". What is 5 - 3?

By the above,

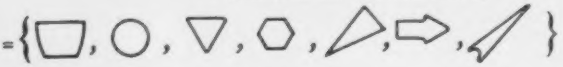
5 - 3 = c if and only if 5 = c + 3.

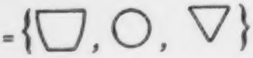
That is to say, 5 - 3 is that number c such that 5 = c + 3; moreover, the number(s) that makes 5 = c + 3 a true statement are the only ones that qualify to be 5 - 3. Since 5 = c + 3 is true only if c is 2, then 5 - 3 must be 2.

VOCABULARY

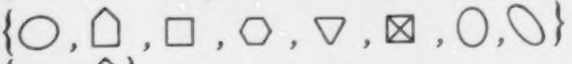
Complement*	Remaining Set*
Difference*	Subtraction*
Inverse Operation*	

EXERCISES - CHAPTER 9

1. A = 

   B = 

   Join to B a set C disjoint from B such that $B \cup C = A$.

2. If A = 

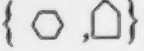
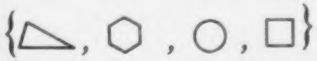
   and B = 

   exhibit A - B.

3. If from a set of 8 members we remove a set of 2 members, how many members does the resulting set have?

4. If A = 

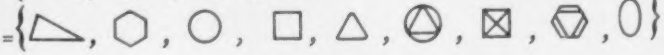
   and C = 

   exhibit B such that $A \cup B = C$. What is N(B)?

5. Show a representation on the number line which illustrates the fact that 10 - 3 = 7. Use the same figure to illustrate the idea that 10 = 7 + 3.

6. Show a representation on the number line which illustrates that the associative property does not hold under the operation of subtraction.

$$(9 - 6) - 3 \neq 9 - (6 - 3)$$

7. What operation is the inverse of adding 7 to any number? What is the inverse of subtracting 8?

8. If A and B are disjoint, illustrate that $(A \cup B) - B = A$. What happens if A and B are not disjoint?

SOLUTIONS FOR PROBLEMS

1. Choose $A = \{0, \triangle, \square, \star, \bigcirc, \mathcal{E}, \boxtimes\}$ with $N(A) = 7$.
Choose $B = \{\star, \bigcirc, \square\}$ which is a subset of A and $N(B) = 3$.
$A \sim B = \{\square, 0, \mathcal{E}, \triangle\}$
By definition, we know that $7 - 3 = N(A - B) = 4$.

2. Choose $A = \{0, \triangle, \square, \star, \bigcirc, \mathcal{E}, \boxtimes\}$ with $N(A) = 7$.
Choose $B = \{a, b, c\}$ with $N(B) = 3$.
Now choose a set C disjoint from both A and B.
$C = \{\Theta, \Xi, \bullet, \Phi\}$ and $N(C) = 4$
so that by matching $(B \cup C)$ with A we can put $B \cup C$ in one-to-one correspondence with A.

$$\begin{array}{rcccccccc} B \cup C = \{ & a, & b, & c, & \Theta, & \Xi, & \bullet, & \Phi & \} \\ & \updownarrow & \updownarrow & \updownarrow & \updownarrow & \updownarrow & \updownarrow & \updownarrow & \\ A = \{ & 0, & \triangle, & \square, & \star, & \bigcirc, & \mathcal{E}, & \boxtimes & \} \end{array}$$

By definition we know that $7 - 3 = N(C) = 4$.

3. By using whole numbers 6, 4, we can illustrate the fact that $a - b = c$ and $a = c + b$ mean the same thing. Thus

$6 - 4 = 2$ because $6 = 2 + 4$
and
$6 - 2 = 4$ because $6 = 4 + 2$.

4. When $a = b$, then $a + b = c$ gives only one subtraction fact; namely $a = c - b$. For example, $3 + 3 = 6$ and $3 = 6 - 3$.

5. a. Inverse property of addition and subtraction
   b. inverse property of addition and subtraction
   c. inverse property of addition and subtraction showing grouping within the parentheses. $30 - 15$ is another name for 15.
   d. identity property of zero for addition (Zero added to any number results in that number.)
   e. identity property of zero for subtraction (Zero subtracted from any number results in that number.)

6. $(5 - 7) + 7$ does not make sense in the present context because $5 - 7$ is not a whole number. For any numbers a and b, $(a - b) + b = a$ if $a \geq b$.

7. To show that $a + (b - a) = b$ if $b > a$ we use the commutative property of addition getting $a + (b - a) = (b - a) + a$, which by the third item in Properties of Subtraction is equal to b.

Chapter 10

DIVISION

DIVISION

In Chapter 8, a rectangular array of a rows with b members in each row was used as a physical model for $a \times b$. From this and from other models, the properties of multiplication for whole numbers were developed. We saw that multiplication of whole numbers has the properties of closure, commutativity, and associativity, and that multiplication is distributive over addition. Also, the numbers 1 and 0 have the special properties that

$$1 \times a = a \times 1 = a, \text{ and}$$
$$0 \times a = a \times 0 = 0.$$

The first three properties exactly parallel the same three properties for addition, and 1 plays a role for multiplication closely corresponding to that of 0 for addition. The similarity in behavior of the two operations leads to the question as to whether there is an operation which bears to multiplication a similar relation as subtraction does to addition; namely, an inverse or undoing operation. The answer to this is the operation called division.

To find the product 4×5, we counted the number of members in a 4 by 5 array or in 4 disjoint sets with 5 members in each set. An associated problem is to start with 20 objects and ask how many disjoint subsets there are in this set if each subset is to have 4 members. In terms of arrays, the question is "if a set of 20 members is arranged 4 to a row, how many rows will there be?" The answer is 5.

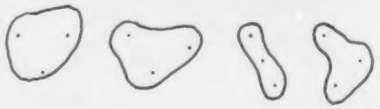
x x x x
x x x x
x x x x
x x x x
x x x x

20 objects arranged 4 to a row.

In many cases there would be no answer to the question. For example, 20 objects arranged 6 to a row does not give an exact number of rows. It is true that ordinarily we do carry out such a division process as 20 divided by 6, obtaining a quotient and a remainder. In speaking of division as an operation in the set of whole numbers, however, the expression "20 divided by 6" is meaningless because it is not a whole number. The process as indicated by $6\overline{)\overset{3}{20}}$, remainder 2, will be more fully developed later when the techniques of division are discussed in detail. It will then be pointed out that for any ordered pair (a, b) with $b \neq 0$, we may develop a division process.

To answer the question, "how many disjoint subsets are there in a set of 20 if each subset is to have 4 members?", we formed an array of 20 objects arranged 4 to a row. When we form this array, we are partitioning the set of 20 into equivalent sets. By partitioning a set, we mean separating it into disjoint subsets. Thus, the fact that a set of 20 may be partitioned into 5 equivalent subsets, each having 4 members, shows us that $20 = 4 \times 5$ and $20 = 5 \times 4$. The number, 5, which is thus assigned to the ordered pair (20, 4) is called the quotient and the operation which produces 5 from (20, 4) is called division. The normal symbol for the operation of division is ÷ . Thus $20 \div 4 = 5$. The partitioning, of course, does not have to be shown as an array. Either diagram below, for example, gives the result of $12 \div 3$.

12 objects, 3 in each row.



Set of 12 objects in disjoint subsets, 3 objects in each subset.

For the ordered pair (20, 6) there is no such whole number that can be attached; nor is there for (5, 15). So, under the operation of division, (20, 6) or (5, 15) are not defined in the set of whole numbers. Division therefore does not have the property of closure in the set of whole numbers. The last case for (5, 15) is simply an example of the fact that in the ordered pair of whole numbers (a, b), if $b > a$, and $a \neq 0$, the operation of division never yields a whole number.

PROBLEMS*

1. Find the whole number attached to each of the following ordered pairs under the operation of division; if there is none, explain.

a. (20, 5)	c. (6, 1)	e. (64, 8)
b. (4, 28)	d. (72, 9)	f. (42, 7)
		g. (47, 7)

2. a. Display an array to show $28 \div 7$.

   b. Illustrate $28 \div 7$ by a partitioning that is other than an array.

By partitioning, we have obtained 5 as the result of $20 \div 4$ because $20 = 5 \times 4$. This is similar to the missing addend approach to subtraction. Here, we say that $a \div b$ is that number c such that $a = c \times b$. That is,

$$a \div b = c \text{ IF AND ONLY IF } a = c \times b.$$

Thus, c is the missing factor of $a = c \times b$ for given numbers a and b, with $b \neq 0$.

DIVISION AS INVERSE

In the same way as subtraction is the inverse of addition, division by a number n may be thought of as the inverse of multiplication by n. Thus,

$$(8 \times 3) \div 3 = 8 \text{ and } (17 \times 4) \div 4 = 17.$$

However, caution must be exercised in thinking about multiplication as the inverse of division because it is true that

$$(15 \div 3) \times 3 = 15, \text{ while } (8 \div 3) \times 3 \text{ is meaningless}$$

since $8 \div 3$ is not a whole number. This is similar to the caution we must exercise in this "doing and undoing" process with subtraction; thus while

$$(15 - 3) + 3 = 15 \text{ is perfectly acceptable,}$$

$$(5 - 13) + 13 \text{ is meaningless}$$

since $(5 - 13)$ is not a whole number. Of course, the restriction will be removed later when the set of whole numbers is extended to include numbers for which $8 \div 3$ and $5 - 3$ have meaning.

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*Solutions for problems in this chapter on page 180.

PROBLEM

3. Tell whether each of the following statements is true or whether it is meaningless for whole numbers.

a. $(3 + 9) - 9 = 3$

b. $(9 + 3) - 9 = 3$

c. $(3 - 9) + 9 = 3$

d. $(3 \times 9) \div 3 = 9$

e. $(3 \div 9) \times 9 = 3$

f. $(9 \times 3) \div 3 = 9$

g. $(9 \div 3) \times 3 = 9$

THE ROLE OF 1 AND 0 IN DIVISION

The operation of division was connected to the operation of multiplication by the statement that

$$a \div b = c \text{ if and only if } a = c \times b, \text{ and } b \neq 0.$$

Since 1 and 0 played special roles in multiplication, it may be appropriate to pay particular attention to the two numbers in division.

If $b = 1$, then we have $a \div 1 = c$ if and only if $a = c \times 1$. Recalling the special property of 1 under multiplication, we have $c \times 1 = c$; hence, a and c represent the same number, and for any whole number a, $a \div 1 = a$. On the other hand, $1 \div b$ is not a whole number unless $b = 1$; if $b \neq 1$ there is no whole number c such that $1 = c \times b$.

In the sense that $a \div 1 = a$, the number 1 acts somewhat like an identity element for division. Unlike the identity element for multiplication in which, for any a, $1 \times a = a \times 1$, the number 1 is limited to acting as an identity element for division only if it is to the right of the symbol $\div$.

Again by the definition of division, we can note that the role of 0 in division. Briefly, its role may be summarized as follows.

$0 \div b = c$ if and only if $0 = c \times b$. For $b \neq 0$, this is true only if $c = 0$. Therefore,

FOR ANY WHOLE NUMBER b SUCH THAT $b \neq 0$, $0 \div b = 0$.

If $b = 0$, we have $0 \div 0 = c$ and $0 = c \times 0$. Since this is true for any number c, the result of $0 \div 0$ is ambiguous; $0 \div 0$ does not specify a unique number, hence

THE OPERATION OF DIVISION IS NOT DEFINED FOR $0 \div 0$.

$a \div 0$ where $a \neq 0$ is still another situation. Since $a \div 0 = c$ if and only if $a = c \times 0$, and $c \times 0 = 0$ for whatever number c, we have a contradiction in terms; we started out with the assumption that $a \neq 0$ and came to the conclusion that $a = 0$. For this reason,

FOR $a \neq 0$, $a \div 0$ IS UNDEFINED.

These last two results together indicate that division by 0 is not defined.

PROBLEM

4. Tell whether each of the following is a whole number, is not a whole number, or cannot be determined; if possible, name the whole number.

   a. $8 \div 4$
   b. $2 \div 4$
   c. $3 \div 3$
   d. $6 \div 0$
   e. $0 \div 132$
   f. $1 \div b$, b is a whole number and $b = 1$
   g. $1 \div b$, b is a whole number and $b \neq 1$.
   h. $a \div b$, a and b are whole numbers and $b > a$.
   i. $0 \div b$, b is a whole number and $b \neq 0$.
   j. $a \div b$, a and b are whole numbers and $a > b$.
   k. $a \div b$, a and b are whole numbers and $a = b$.

PROPERTIES OF DIVISION

Many examples may be given to show that the whole numbers are not closed under division. For example, while $6 \div 3 = 2$, $3 \div 6$ is not a whole number. These same two examples show that $6 \div 3 \neq 3 \div 6$, hence the operation is not commutative. To see that division is not associative, again many examples may be produced. We need only one example, and such an example is the following:

$$(12 \div 6) \div 2 = 2 \div 2 = 1, \text{ but}$$

$$12 \div (6 \div 2) = 12 \div 3 = 4.$$

The different results obtained for $(12 \div 6) \div 2$ on the one hand, and for $12 \div (6 \div 2)$ on the other, shows that, in general, it is not true that $(a \div b) \div c$ equals $a \div (b \div c)$.

So far, division with respect to whole numbers has revealed itself as an operation that does not have the properties of closure, commutativity and associativity. Furthermore, division by 0 is impossible. To free ourselves from the impression that not much can be said about this operation, we need to consider only the important notion that division by b is the inverse of the operation of multiplication by b. That is, $(a \times b) \div b = a$, provided, of course, $b \neq 0$.

PROBLEMS

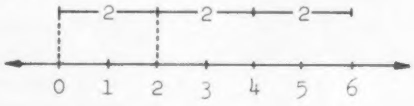
5. For which of the following is it true that $(a \div b) \div c = a \div (b \div c)$?

a. $4 \div 2 \div 2$	e. $9 \div 9 \div 1$
b. $4 \div 2 \div 1$	f. $9 \div 3 \div 1$
c. $24 \div 6 \div 2$	g. $0 \div 9 \div 3$
d. $0 \div 5 \div 1$	

6. From the results of the preceding exercises, under what conditions will $(a \div b) \div c = a \div (b \div c)$?

DIVISION USING THE NUMBER LINE

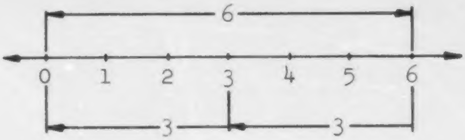
We can illustrate division using the number line by partitioning a segment into congruent subsegments. For example, to illustrate $6 \div 3$, we can partition a 6 unit segment into 3 congruent subsegments, each of which



is congruent to the segment from 0 to 2. Thus, this partition conveys the concept $6 \div 3 = 2$. Clearly, this is associated with the representation of multiplication on the line in which three 2 unit arrows or 2 unit segments are abutted, resulting in a 6 unit arrow or a 6 unit segment. The association may be thought of as: one operation is the inverse of the other, or, from the point of view that

$$6 \div 3 = 2 \text{ if and only if } 6 = 2 \times 3.$$

Another method of illustrating division on the number line is related to considering division in terms of repeated subtraction. This concept will be discussed in further detail in Chapter 15 when the division techniques are discussed. We can indicate here, however, this use of the number line in order to compare with the use shown above. Beginning with 6,



we ask: How many times can 3 be subtracted? Corresponding to this, we can show division using the number line as in the above figure. In this case, since subtraction is performed twice, $6 \div 3 = 2$.

PROBLEM

7. a. Show by partitioning a segment on the number line that $10 \div 2 = 5$.

   b. Show by partitioning a segment on the number line that $5 \div 2$ does not yield a whole number.

COMPOSITE NUMBERS

Rectangular arrays form the basis for what used to be known as the "rectangular numbers" by the ancient Greeks. If a number n can be presented as other than a 1 by n array, then the n is said to be a rectangular number. For example, 6 may be represented by a 2 by 3 array, so 6 is a rectangular number. Now we call such a number a composite number; $6 = 2 \times 3$, so 6 is "composed" of 2 and 3. 12 is also a composite number; either a 3 by 4 rectangular array or a 2 by 6 rectangular array may be used as a model for the composition of 12. However, $2 \times 2 \times 3$ also shows how 12 may be composed. It is true that if a whole number n may be "decomposed" into more than two factors (other than 1 and n), then it can be decomposed into two factors other than 1 and n. Hence, such a number would be considered also a rectangular number. It is simply that thinking in terms of the composition puts the focus more on analyzing the number than thinking in terms of rectangular arrays that can be formed.

Since $12 = 3 \times 4$, we have regarded 3 and 4 as factors of 12. As we have noted, there are other factors of 12. For example, 2 is a factor of 12 because there is a whole number whose product with 2 is 12. That is, 2 is a factor of 12 because 12 is 2 times a whole number; in this case, the whole number is 6. This automatically qualifies 6 to be also a factor of 12. A complete list of factors of 12 may be catalogued as follows:

$12 = 1 \times 12$, so 1 and 12 are factors of 12;
$12 = 2 \times 6$, so 2 and 6 are factors of 12;
$12 = 3 \times 4$, so 3 and 4 are factors of 12;
$12 = 4 \times 3$, so 4 and 3 are factors of 12;
$12 = 6 \times 2$, so 6 and 2 are factors of 12;
$12 = 12 \times 1$, so 12 and 1 are factors of 12;

Thus, 12 has 1, 2, 3, 4, 6, and 12 as factors. 5 is not a factor of 12 because there is no whole number n such that the mathematical sentence

$$12 = 5 \times n$$

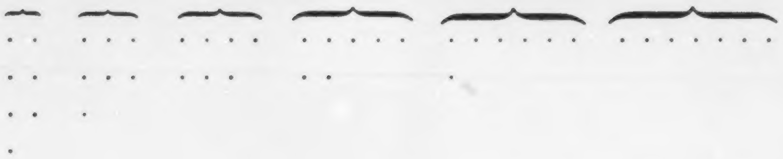
is true. Neither are 7,8,9,10,11, and any whole number greater than 12 factors of 12. (Notice that the last three statements in the display give no information on factors that was not contained in the first three statements and we could have done without them.)

It is clear that since $n = 1 \times n$, any whole number n has 1 and n as factors. However, there are many whole numbers for which these are the only factors. For example 1 and 5 are the only factors of 5; 1 and 7 are the only factors of 7; and 1 and 13 are the only factors of 13; and so on. Such numbers will be of interest for us and are specially designated.

ANY WHOLE NUMBER THAT HAS EXACTLY TWO DIFFERENT WHOLE NUMBER FACTORS (NAMELY ITSELF AND 1) IS A PRIME NUMBER.

Note that this definiton excludes 1 from the set of prime numbers because 1 does not have two different factors. It also excludes 0 from the set of primes since $0 = 0 \times n$ for any whole number n; any whole number is a factor of 0. In essence, the prime numbers are those that can only be associated with a 1 by n array (for $n \neq 1$). For example, let us consider an array for 7. Placing two objects in each row, we can complete an array with 6 objects; the seventh object makes the array incomplete.

Similarly,

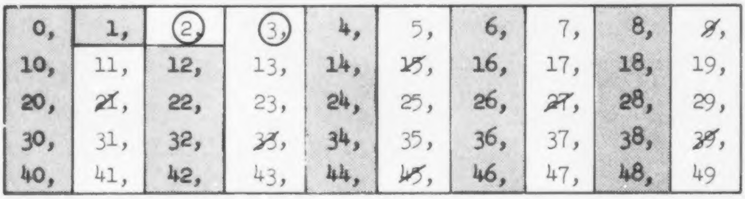


3, 4, 5, or 6 objects in a row induce incomplete arrays with 7 objects.

All whole numbers greater than 1 may now be classified according to whether they are prime or composite. Over 2,000 years ago, the mathematician Eratosthenes devised an easy and straightforward method for sorting prime numbers from a list of whole numbers. To find all the prime numbers less than 50, for example, the whole numbers from 0 through 49 are listed as below. 0 and 1 are crossed out since they are not primes. 2 is a prime, but every other even number has 2 as a factor, so all even numbers greater than 2 are crossed out.

~~0~~, ~~1~~, 2, 3, ~~4~~, 5, ~~6~~, 7, ~~8~~, 9,
~~10~~, 11, ~~12~~, 13, ~~14~~, 15, ~~16~~, 17, ~~18~~, 19,
~~20~~, 21, ~~22~~, 23, ~~24~~, 25, ~~26~~, 27, ~~28~~, 29,
~~30~~, 31, ~~32~~, 33, ~~34~~, 35, ~~36~~, 37, ~~38~~, 39,
~~40~~, 41, ~~42~~, 43, ~~44~~, 45, ~~46~~, 47, ~~48~~, 49

Continuing with this, 3 is "saved" and 3 × 2, 3 × 3, 3 × 4, ..., are "eliminated"; that is, all "multiples" of 3 greater than 3 × 1 are eliminated.



0,	1,	②,	③,	4,	5,	6,	7,	8,	~~9~~,
10,	11,	12,	13,	14,	~~15~~,	16,	17,	18,	19,
20,	~~21~~,	22,	23,	24,	25,	26,	~~27~~,	28,	29,
30,	31,	32,	~~33~~,	34,	35,	36,	37,	38,	~~39~~,
40,	41,	42,	43,	44,	~~45~~,	46,	47,	48,	49

In this second chart, the numerals that are shaded represent numbers that are "eliminated" after the screening as "multiples" of 2 (1 is "eliminated" before the screening). The slash marks indicate screening as "multiples" of 3, and the numbers that are "saved" are identified by circles. By now, 4 has been eliminated because it is a multiple of 2; 5 is next saved and all other multiples of 5 eliminated and so on. Thus, eventually, we arrive at the set of all prime numbers less than 50:

$\{2, 3, 5, 7, 11, 13, 17, 19, 23, 29, 31, 37, 41, 43, 47\}$.

It can be shown that this screening process needs not be carried beyond 7 for prime numbers less than 50 since $49 = 7 \times 7$. If 49 is the product of two whole numbers a and b, and one of these is greater than 7, then the other must be less than 7. This tells us that any factor greater than 7 would have been eliminated when its companion factor (which is less than 7) was considered.

PROBLEMS

8. Express each of the following numbers as products of two factors in several ways, or indicate that it is impossible to do so.

   a. 18 c. 30

   b. 6 d. 11

9. List all the numbers that could be called "factors"

   a. of the number 30,

   b. of the number 19,

   c. of the number 24.

FACTORING COMPOSITE NUMBERS.

A prime number can be expressed as the product of counting numbers in one and only one way, namely the product of 1 and itself. Thus

$$\begin{aligned} 3 &= 3 \times 1, \\ 5 &= 5 \times 1, \\ 7 &= 7 \times 1, \\ 11 &= 11 \times 1. \end{aligned}$$

A composite number has more than one factor expression. For example, some factor expressions of 24 are

$$\begin{aligned} 24 &= 1 \times 24 \\ &= 2 \times 12 \\ &= 3 \times 8 \\ &= 4 \times 6 \\ &= 4 \times 2 \times 3 \\ &= 2 \times 2 \times 6 \\ &= 2 \times 2 \times 2 \times 3. \end{aligned}$$

Notice in the expression $2 \times 2 \times 2 \times 3$ all the factors are prime numbers. Because of this it is called the *complete factorization* or the *prime factorization* of 24. It expresses 24 as a product of prime numbers.

Every composite number can be factored; that is, it can be written as the product of at least two factors each of which is less than the number itself. If one or more of these factors is a composite number, it can be written as the product of still smaller factors. This process cannot go on indefinitely since the factors, which are counting numbers, are getting smaller, and the smallest counting number is 1. Eventually we must come to a factor expression each of whose factors is a prime. For example

$$\begin{aligned} 360 &= 6 \times 60 \\ &= 2 \times 3 \times 60 \\ &= 2 \times 3 \times 5 \times 12 \\ &= 2 \times 3 \times 5 \times 2 \times 6 \\ &= 2 \times 3 \times 5 \times 2 \times 2 \times 3 \end{aligned}$$

$$\begin{aligned} 360 &= 9 \times 40 \\ &= 3 \times 3 \times 40 \\ &= 3 \times 3 \times 5 \times 8 \\ &= 3 \times 3 \times 5 \times 2 \times 4 \\ &= 3 \times 3 \times 5 \times 2 \times 2 \times 2. \end{aligned}$$

$$\begin{aligned} 360 &= 12 \times 30 \\ &= 2 \times 6 \times 30 \\ &= 2 \times 2 \times 3 \times 30 \\ &= 2 \times 2 \times 3 \times 3 \times 10 \\ &= 2 \times 2 \times 3 \times 3 \times 2 \times 5. \end{aligned}$$

Notice that although in each case above we started with a different pair of factors, the complete factorization was the same except for the order in which the prime factors were written. This is always true. Every composite number can be written as the product of primes in one and only one way except for the order in which the prime factors are written.

PROBLEM

10. Find the prime factorization of each of the following.

a.	8	d.	160
b.	27	e.	144
c.	24	f.	210

GREATEST COMMON FACTOR

Let us consider the numbers 8 and 12. We see that both 8 and 12 are even numbers, hence they both have a factor 2. Because 2 is a factor of both 8 and 12, we say it is a common factor of the numbers.

All whole numbers are divisible by 1, hence 1 is a common factor of all whole numbers. Therefore when we look for common factors of several numbers, we need only look for numbers greater than 1, because we already know that 1 is one of the common factors.

Let us ask ourselves what factors are common factors of 8 and 12.

$8 = 1 \times 8$ $\qquad$ $12 = 1 \times 12$

$8 = 2 \times 4$ $\qquad$ $12 = 2 \times 6$

$8 = 2 \times 2 \times 2$ $\qquad$ $12 = 3 \times 4$

$12 = 3 \times 2 \times 2$

The set of all factors of 8 is

$$\{1, 2, 4, 8\}.$$

The set of all factors of 12 is

$$\{1, 2, 3, 4, 6, 12\}.$$

The set of common factors of 8 and 12 is the intersection of the two sets above.

$$\{1, 2, 4, 8\} \cap \{1, 2, 3, 4, 6, 12\} = \{1, 2, 4\}.$$

Hence, the common factors of 8 and 12 are 1, 2, 4.

Do the numbers 5 and 8 have any common factors other than 1? The set of all factors of 5 is

$$\{1, 5\}.$$

The set of all factors of 8 is

$$\{1, 2, 4, 8\}.$$

The intersection set of these two sets is

$$\{1\}.$$

Hence 5 and 8 have only one common factor, 1. The answer to our question then is "The numbers 5 and 8 do not have any common factors other than 1".

Some sets of numbers have many common factors and some sets have only 1 as a common factor.

The greatest element in the set of common factors of several numbers is called the greatest common factor of these numbers. We see then that 4 is the greatest common factor of 8 and 12; 1 is the greatest common factor of 5 and 8.

Writing the set of all factors of a number is sometimes troublesome, especially if the number has many factors. An easier way to find the greatest common factor of several numbers is the use of their complete factorization.

Suppose we wish to find the greatest common factor of 36, 42, and 72. We find the complete factorization of the numbers:

$$36 = 2 \times 2 \times 3 \times 3 = (2 \times 3) \times 2 \times 3$$

$$42 = 2 \times 3 \times 7 = (2 \times 3) \times 7$$

$$72 = 2 \times 2 \times 2 \times 3 \times 3 = (2 \times 3) \times 2 \times 2 \times 3.$$

Notice that each number has 2 as a common factor and 3 as a common factor. Hence $2 \times 3 = 6$ is a common factor of them. All the common factors of 36, 42, and 72 are 1, 2, 3 and 6. The greatest common factor of these numbers is $2 \times 3 = 6$.

PROBLEM

1. Find the greatest common factor of the sets of numbers below
   a. 6, 8
   b. 3, 8, 12
   c. 24, 16
   d. 36, 48, 56

APPLICATION TO TEACHING

The topics of factors, composite numbers, and prime numbers will not be presented until the second grade. A start on this is given in the first grade when we count by twos. Of course, in terms of multiples, the even numbers are simply the multiples of 2. Similarly, multiples of 3 are the entries in the "3 times" table, and so on.

We have noted that since 3 is a factor of 12, we can say that 12 is a multiple of 3. Both factor and multiple originate from the same concept: there is a whole number n such that $12 = 3 \times n$. A multiple is viewed from the standpoint of the number being composed; a factor is viewed from the standpoint of a number going into the composition as a "building block". Beginning in Grade 5, the children will be introduced to the Fundamental Theorem of Arithmetic - when a whole number is "decomposed" into the primitive building blocks of prime numbers, this decomposition will be revealed as unique; that is, a whole number is made up of one and only one set of primitive blocks which we call the primes. At that time, the children will be taught the "complete factorization" of a whole number (or, the prime decomposition). Complete factorization is a natural lead-in to a corresponding factorization in algebra, which yields, among other things, solutions to algebraic equations.

QUESTION

"Doesn't $6 \div 3 = 2$ show that division is closed in the set of whole numbers?"

The statement, $6 \div 3 = 2$, asserts that there is a whole number that answers the question "What is 6 divided by 3?" To say that division is closed in the set of whole numbers means that without exception, for any whole numbers a and b, $b \neq 0$, we must be able to find a whole number represented by $a \div b$. Since examples can be found to deny that it is always true that $a \div b$ results in a whole number, we cannot say that division is closed in this set. For instance, $3 \div 6$ is not a whole number. One example cannot be used to prove a general statement.

VOCABULARY

Common Factor
Complete Factorization *
Composite Number *
Division *
Factor *
Greatest Common Factor
Inverse Operation *
Prime Factorization*
Prime Number *
Quotient *

EXERCISES - CHAPTER 10

1. Rewrite each mathematical sentence below as a division sentence. Find the unknown factor.

   a. $n \times 5 = 20$
   b. $p \times 4 = 28$
   c. $n \times 1 = 6$
   d. $n \times 9 = 72$
   e. $n \times 8 = 64$
   f. $q \times 0 = 0$

2. Tell whether each of the following is more readily visualized by a rectangular array of 7 rows or by disjoint subsets with 7 in each subset.

   a. 42 pieces of candy are to be divided equally among 7 children.
   b. 42 pieces of candy are to be packaged 7 pieces to a package.

3. A marching band always forms an array when it marches. The leader likes to use many different formations. Aside from the leader, the band has 59 members. The leader is trying very hard to find one more member. Why?

4. Does division have the commutative property? Give an example to substantiate your answer.

5. Express each of the following numbers as a product of two smaller numbers or indicate that it is impossible to do this:

   a. 12
   b. 36
   c. 31
   d. 7
   e. 8
   f. 11
   g. 35
   h. 5
   i. 39
   j. 42
   k. 6
   l. 41
   m. 82
   n. 95

6. Factor each number below completely

   a. 16
   b. 21
   c. 63
   d. 90
   e. 144
   f. 132

7. Find the greatest common factor of each set of numbers below.

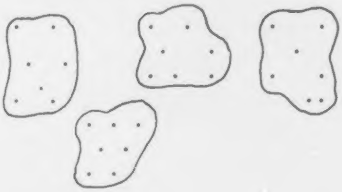
a. 2, 3
b. 15, 8
c. 6, 16
d. 3, 12, 15
e. 3, 8, 30
f. 12, 16
g. 9, 33, 21
h. 8, 16, 56

SOLUTIONS FOR PROBLEMS

1. a. 4
b. None; $28 > 4$ and $4 \neq 0$.
c. 6
d. 8
e. 8
f. 6
g. None; there is no 7 row array of 47 members.

2. a.
. . . .
. . . .
. . . .
. . . .

b.



3. a. True
b. True
c. Meaningless
d. True
e. Meaningless
f. True
g. True

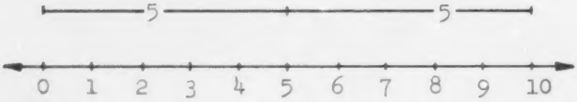
4. a. Whole number; 2
b. Not a whole number
c. Whole number; 1
d. Not a whole number
e. Whole number; 0
f. Whole number; 1
g. Cannot be determined: meaningless if $b = 0$; not a whole number if $b > 1$.
h. Cannot be determined: zero if $a = 0$; not a whole number if $a \neq 0$.
i. Whole number; 0
j. Cannot be determined: meaningless if $b = 0$; whole number a if $b = 1$; whole number if $b > 1$ and b is a factor of a; not a whole number if $b > 1$ and b is not a factor of a.

k. Cannot be determined; undefined if $a = b = 0$; the whole number 1 if $a = b \neq 0$.

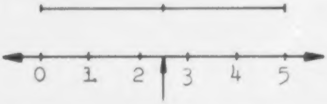
5. a. False
   b. True
   c. False
   d. True
   e. True
   f. True
   g. True

6. If $a = 0$, or $c = 1$, or both $a = 0$ and $c = 1$.

7. a.



b.



The coordinate of this point is not a whole number.

8. a. 3×6, 2×9, 1×18 (or 6×3, 9×2, etc.)
   b. 2×3, 1×6
   c. 2×15, 5×6, 3×10, 1×30
   d. 1×11 and 11×1 are the only such factorizations and they are not essentially different.

9. a. 1, 2, 3, 5, 6, 10, 15, and 30
   In more formal terms, the set of factors of $30 = \{1,2,3,5,6,10,15,30\}$
   b. 1 and 19
   c. The set of factors of $24 = \{1,2,3,4,6,8,12,24\}$

10. a. $2 \times 2 \times 2$
    b. $3 \times 3 \times 3$
    c. $2 \times 2 \times 2 \times 3$
    d. $2 \times 2 \times 2 \times 2 \times 2 \times 2 \times 5$
    e. $2 \times 2 \times 2 \times 2 \times 3 \times 3$
    f. $2 \times 3 \times 5 \times 7$

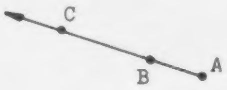
11. a. 2
    b. 1
    c. 8
    d. 4

Chapter 11

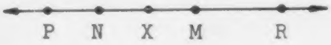
ELEMENTS OF GEOMETRY

RAYS

In Chapter 5 we introduced some geometric concepts using physical objects such as blocks, balls, boxes, cans, and ice cream cones as models. From these models, we conceived idealized sets of points such as rectangular solids, spheres, cylinders, cones, segments, and so on. There is another set of points which is important in geometry. This geometric configuration can be formed by extending a line segment in one direction only. This figure is called a ray. A ray is indicated below.



The ray shown above is formed by extending $\overline{AB}$ through B or $\overline{AC}$ through C. The notation for this ray is $\overrightarrow{AB}$ or $\overrightarrow{AC}$. In contrast with the notation for segments and lines, in naming rays the order of points is significant. A ray has one endpoint, and it is named first. The second letter can name any other point in the ray. As indicated below, $\overrightarrow{MR}$ and $\overrightarrow{NR}$ are not equal sets. There are common points in the two sets. However, point X is in $\overrightarrow{MP}$ but is not a point in $\overrightarrow{NP}$.



Note that the arrow in the nomenclature $\overrightarrow{MP}$ designates which is the endpoint of the ray; it is not the intention to convey the orientation of the ray as it appears. In fact, it would be impossible to orient the arrows in conformity with all possible orientations of the ray.

PROBLEMS*

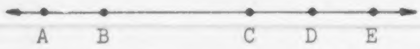
1. Represent $\overleftrightarrow{PR}$ and show Q between P and R. Which of the following denote the same ray?
   $\overrightarrow{PQ}$, $\overrightarrow{QP}$, $\overrightarrow{QR}$, $\overrightarrow{PR}$, $\overrightarrow{RP}$, $\overrightarrow{RQ}$

---

*Solutions for problems in this chapter are on page 196.

2. a. What is the implication of the statement $\overrightarrow{AB} = \overrightarrow{CB}$?
   b. Does $\overrightarrow{BA} = \overrightarrow{BC}$ have a similar conotation?

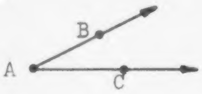
3. Referring to the drawing below, rename the sets in simple notation.



a. Union of $\overline{BC}$, $\overline{CD}$ and $\overline{DE}$
b. Intersection of $\overline{AB}$ and $\overline{BC}$.
c. Intersection of $\overrightarrow{CA}$ and $\overrightarrow{ED}$.
d. Intersection of $\overrightarrow{CD}$ and $\overrightarrow{DC}$.
e. Union of $\overrightarrow{EA}$ and $\overrightarrow{BC}$.

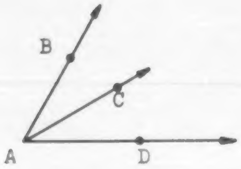
ANGLE

Another fundamental geometric figure recognized in many familiar shapes is an angle. The formal definition is: an angle is the union of two rays which have a common endpoint but which are not subsets of the same line.



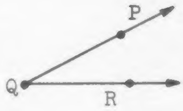
The example shown is the union of $\overrightarrow{AB}$ and $\overrightarrow{AC}$. Their common endpoint is said to be the vertex of the angle. Recall that vertex also applies to geometric solids and their faces. In each case it is the intersection of appropriate edges. Similarly, here, the vertex of an angle is the intersection of the two sets of points in the rays. The rays are called the sides of the angle.

Our angle is denoted by $\angle BAC$ or $\angle CAB$, where the middle letter identifies the vertex. The other two letters name one point distinct from the vertex on each of the two sides. Often, simply $\angle A$ will be written instead of $\angle BAC$. This notation cannot be used if more than one angle is drawn at vertex A.



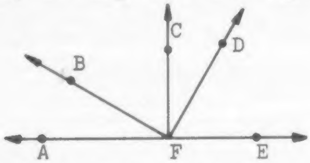
It would not be clear by $\angle A$ which of $\angle BAC$, $\angle CAD$, or $\angle BAD$ were meant in the figure above.

PROBLEMS



4. a. Name in three ways the angle shown.
   b. Identify the sides of this angle.

5. Identify all angles in the figure below.



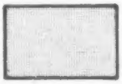
6. Can $\overrightarrow{VX}$ and $\overrightarrow{WX}$ be sides of an angle?

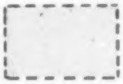
REGIONS

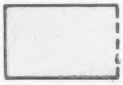
Since a polygon is a simple closed curve, it is the set of points on the curve. These points should be distinguished from the set of points enclosed by the curve which we call the interior; the two sets are disjoint. A circle is also a simple closed curve, and it also has an interior. The union of a simple closed curve and its interior is called a region. We refer to a triangular region, rectangular region, polygonal region, or circular region, etc., indicating that the simple closed curve is a triangle, rectangle, polygon, circle, etc.

To denote a plane region in a diagram, the interior of the simple closed curve is usually shaded. To denote the interior only, the interior is shaded, but the polygon is drawn in dashed outline as is shown in the figures below.

rectangle


rectangular region

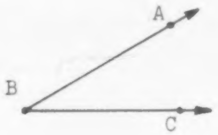

interior of
rectangle

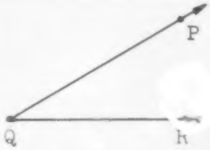

union of interior
of rectangle and
part of rectangle

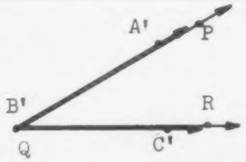
CONGRUENT ANGLES

In Chapter 5 we defined two geometric figures as congruent if one is an exact copy of the other.

Suppose we are given two angles, ∠ABC and ∠PQR, and we wish to find out if they are congruent. We make a tracing of

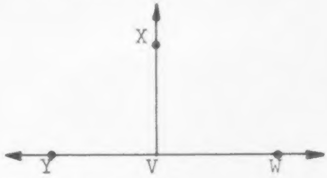






∠ABC, say ∠A'B'C'. We now place the tracing on ∠PQR such that ray $\overrightarrow{B'C'}$ falls on $\overrightarrow{QR}$ and B' falls on Q. (This is shown above at the right.) Now if $\overrightarrow{B'A'}$ falls on $\overrightarrow{QP}$ we say that ∠ABC is congruent to ∠PQR.

A special angle that makes frequent appearances in mathematics is a right angle. No formal defintion is given at this time. Instead, we will describe what is meant by a right angle in much the same way that you will convey the concept to your students.

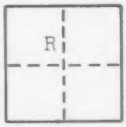


The above drawing represents two right angles, ∠YVX and ∠WVX. The angles are congruent, and the union of a side of one and a side of the other is a line.

If a piece of paper were folded twice, as the drawing below indicates, and it were then unfolded, the creases suggest segments of two lines whose intersection is the point R. Thus, R is the vertex of four right angles whose sides are the extensions of appropriate pairs of creases.

Fold 1

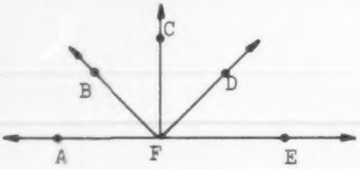
Fold 2



Unfolded with creases dotted

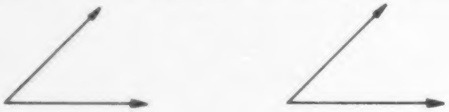
PROBLEMS

7. Identify all angles which appear to be right angles.

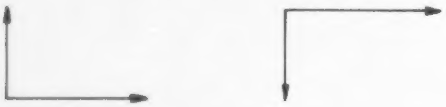


8. Which of the following pairs of angles are congruent?

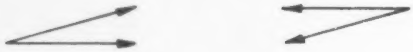
a.



b.



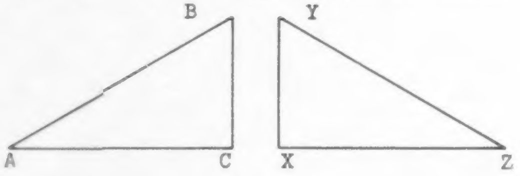
c.



CONGRUENT REGIONS

We have discussed congruent segments and congruent angles. Now we shall discuss congruent regions.

Two geometric regions are congruent if one is an exact copy of the other. Suppose we have the two triangles pictured below.



First we make a tracing of triangle ABC. Now we cut along the boundary. We place this tracing on triangle ZYX in any way that does not distort the region. If the tracing fits exactly the second region, we say the two regions are congruent and the boundaries are congruent.

To summarize, to decide whether or not two regions are congruent:

(1) We make a tracing of the boundary of one region.

(2) We try to match this tracing to the other region.

(3) If the tracing matches the second region with no distortion to either, then the boundaries and the regions are congruent.

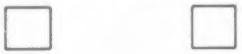
The movable copy is needed because the geometric figures to be compared are sets of points, and as such, have fixed locations. Clearly, we cannot continue this matching process too long. A copy of a solid may not be matched and fitted into another solid. A more refined concept of congruence is not attempted until the children study geometry from a more formal standpoint.

In the light of congruence, we may restate the requirements of special geometric figures. For example, any two edges of a cube are congruent segments, and any two faces of a cube are congruent regions. Similarly, we can note the congruent sides of a parallelogram and so on.

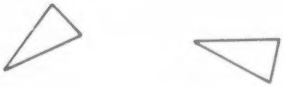
PROBLEM

9. Which of the following pairs of regions are congruent?

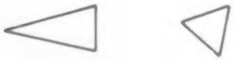
a.



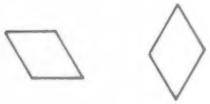
b.



c.



d.



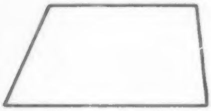
CLASSIFICATION OF POLYGONS

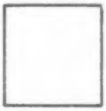
A polygon is a simple closed curve that is a union of line segments. If it is a union of three line segments, it is a triangle; of four line segments, a quadrilateral; of five segments, a pentagon; of six segments, a hexagon; and so on.

The segments which form the polygon are called it *sides*. The endpoints of the sides are the *vertices* of the polygon. Note that each vertex is a common endpoint of two sides. Also, the number of vertices is the same as the number of sides.

QUADRILATERALS

A *quadrilateral* is a polygon of four sides. The figures below are all quadrilaterals.





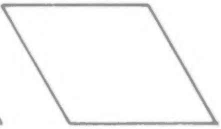
Rectangles are special kinds of quadrilaterals. All the angles* of a rectangle are congruent. *Squares*, in turn, are special kinds of rectangles. All sides of a square are congruent. Thus, in the family

square

rectangle

of quadrilaterals, subfamilies are identified. The rectangles constitute a subfamily of the quadrilaterals and the squares constitute a subfamily of the rectangles. Another subfamily of the quadrilaterals are the *parallelograms*. Their opposite sides are segments of lines which are on the same plane and which do not intersect. As rectangles also possess this characteristic, rectangles are a subfamily of parallelograms. Another subfamily of the parallelograms are the *rhombi* (singular: rhombus). Each side of a rhombus is congruent to each other side. So a square is both a special kind of a rectangle and a special kind of a rhombus.

parallelogram



rhombus

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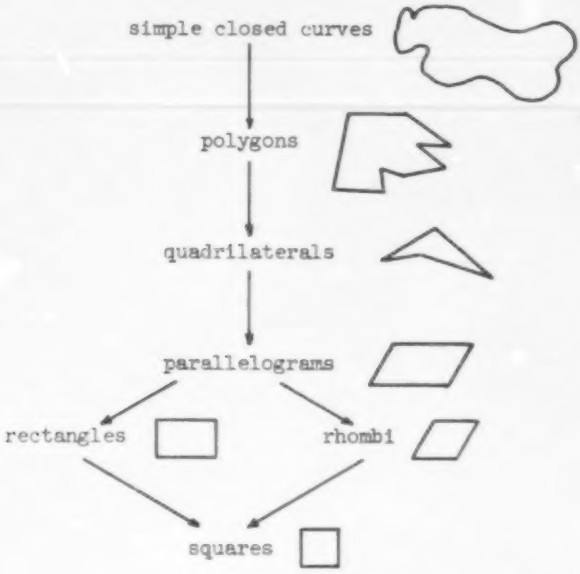
*The term "angle of a polygon" at a particular vertex is a language of convenience to mean the angle having that vertex and such that the particular sides of the polygon belong to the rays of the angle.

By this kind of classification, we get a generic chain that may be indicated by the following diagram.


simple closed curves
polygons
quadrilaterals
parallelograms
rectangles
rhombi
squares

TRIANGLES

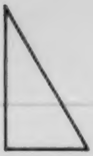
A *triangle* is a polygon of three sides. A triangle may also be defined as a set of three points, not all on the same line, and the three line segments joining these three points as endpoints.

There are three special triangles which shall be of special interest to us. They are the equilateral, the isosceles and the right triangles.

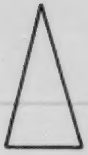
An *equilateral* triangle is a triangle each of whose sides is congruent to the others. In other words, an equilateral triangle has three congruent sides.

An *isosceles* triangle is a triangle with at least two of its sides congruent. So, every equilateral triangle is also an isosceles triangle.

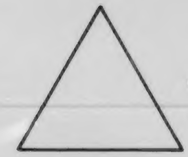
A *right triangle* is a triangle one of whose angles is a right angle.



right triangle



isosceles
triangles



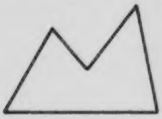
equilateral
triangle

A right triangle may or may not be isosceles; but it cannot be equilateral.

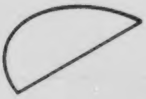
PROBLEMS

10. Which figures pictured below are polygons?

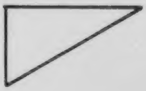
a.



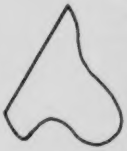
b.



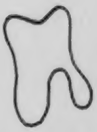
c.



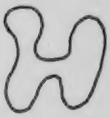
d.



e.

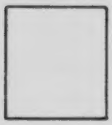


f.

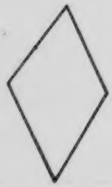


11. Which figures pictured below are quadrilaterals?

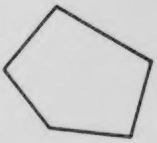
a.



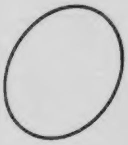
b.



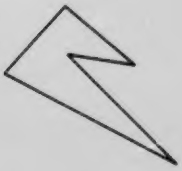
c.



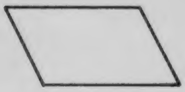
d.



e.



f.



12. Which of the following are true statements?

    a. Every square is a rectangle.
    b. All right triangles are quadrilaterals.
    c. All equilateral triangles are isosceles triangles.
    d. A parallelogram is a rectangle.
    e. A square is a polygon.

APPLICATIONS TO TEACHING

Geometric configurations are sets of points or unions of such sets. A point is a set with a single member. A segment is the union of single member sets. The union of two points is a set and thus, the two points constitute a geometric configuration; so do a point and a curve; and so on. From the union of certain segments or curves, we obtain such familiar figures as triangles, rectangles, circles, pyramids, cones, prisms, and spheres.

The sets of geometric objects that children have to manipulate are sets of three-dimensional objects. These are the concrete objects which provide children with experiences from which they can abstract the mathematical concepts. For this reason, we begin with models of solids. From the models, we identify faces, edges, and vertices. Once identified, we can use these primitive elements to construct other geometric figures. For example, "skeletons" of pyramids and prisms are unions of certain line segments.

Eventually, other kinds of primitive elements will be introduced to serve as building blocks for various geometric figures. These building blocks will be objects called simplexes. They include figures such as segments, triangular regions, and triangular pyramids. The configurations formed by the union of such elements are the complexes such as triangles, and "skeletons" of pyramids. For the study of the complexes, what can be learned about simplexes will be extremely helpful, although not all problems about complexes can be answered by relating complexes to the building blocks. Moreover, certain kinds of complexes give rise to special sets of points called convex sets which play a significant role in the branch of mathematics called linear programming. Linear programming has many applications in business and in the physical and social sciences. The contact which the children at this level have with simplexes and complexes are mainly in terms of polygons and other simple closed curves or solids consisting of edges. Such experiences will form a basis for future experiences in mathematics.

Of particular interest are complexes that are closed figures. Such complexes may specify where solutions to certain existing problems may be found. Closed figures do not have points that can be designated as the initial point and the endpoint. A circle, an oval, a triangle, a figure-eight, the surface of a rectangular box are all examples of closed figures. For children, the approach to closed figures is entirely geometric. It must be emphasized that any closed figure that does not lie in a plane is called a "solid", even though it is hollow. For example, a rectangular "box" consisting only of the faces is a "solid"; the "skeleton" of a rectangular box is a "solid".

It is a good idea to display a set of wooden models that are not too small and encourage the children to examine and handle them for several days before beginning the chapter, "Recognizing Geometric Figures". Tracings of the faces of the solids may be made on a large sheet of paper and displayed so that the children may match a face of a solid to its tracing. Wire or stick models of polygons whose sides are congruent to edges of solids may be used for the same purpose. Matching pictures of solids with the appropriate models should prove useful in helping the children to visualize drawings of 3-dimensional solids. Most pupils seem to be interested in finding objects at home which qualify as cylinders and rectangular boxes and so on.

Solid figures may be identified as blocks, boxes, or balls. For example, a triangular pyramid may be referred to as a block with triangular faces; but it would not be appropriate to identify a ball as a circle or a rectangular prism as a rectangle. Basic distinctions to be made for the children at this time are:

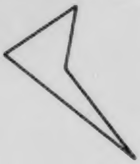
straight edge vs. rounded edge;
flat region vs. rounded region;
flat figure vs. solid figure.

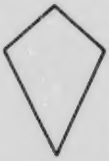
We have stated that in the study of geometry, each of the following objects, a point, a line, and a plane, may be regarded as a primitive element. By these, we can define other geometric objects. Likewise, a 3-space may serve as a primitive element, and it is from this standpoint that we consider points, lines, planes, spaces, as elements of geometry.

QUESTION

"What is meant by saying that a rectangle is a special kind of quadrilateral?"

A quadrilateral is a polygon having exactly four sides. Thus, any of the following represents a quadrilateral:





It can be seen that, of these, a rectangle qualifies to be a quadrilateral; it is a four-sided polygon. However, it distinguishes itself by the special additional requirements having all angles that are congruent. Note too, that a square fulfills all requirements for a rectangle; being a polygon, having four sides, and having angles that are all congruent to each other. The square, however, has the additional requirement of having all sides that are congruent. By the same token that a rectangle is a special kind of a quadrilateral, a square is then a special kind of a rectangle.

VOCABULARY

Angle*
Circular Region
Congruent Regions
Equilateral Triangle *
Geometric Configuration
Isosceles Triangle *
Parallelogram
Plane Region *
Polygon *
Polygonal Region
Quadrilateral *
Ray *
Rectangle *

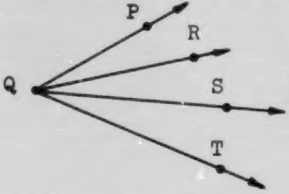
Rectangular Region
Region *
Rhombus
Right Angle *
Right Triangle*
Side of an Angle *
Side of a Polygon *
Square *
Triangle *
Triangular Region
Vertex of an Angle*
Vertex of a Polygon *

EXERCISES - CHAPTER 11

1. If the union of two rays, $\overrightarrow{AB}$ and $\overrightarrow{CD}$, is a line what will the intersection of $\overrightarrow{AB}$ and $\overrightarrow{CD}$ be?

2. Explain the differences between AB, $\overrightarrow{AB}$ and $\overleftrightarrow{AB}$.

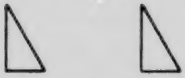
3. Name all the angles shown on the picture below.



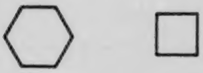
4. Which of the following statements are true?
   a. All rectangles are polygons.
   b. All quadrilaterals are rectangles.
   c. All rectangles are squares.
   d. All parallelograms are polygons.
   e. Polygons are simple closed curves.
   f. All isosceles triangles are polygons.

5. Which of the following pairs of figures are congruent?

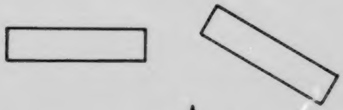
a.



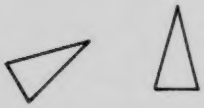
b.



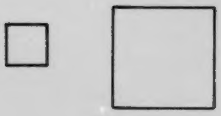
c.



d.

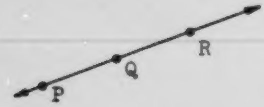


e.



SOLUTIONS FOR PROBLEMS

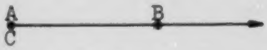
1.



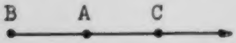
$\overrightarrow{PQ}$ and $\overrightarrow{PR}$ are the same ray or $\overrightarrow{PQ} = \overrightarrow{PR}$

$\overrightarrow{RP}$ and $\overrightarrow{RQ}$ are the same ray or $\overrightarrow{RP} = \overrightarrow{RQ}$

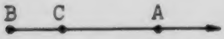
2. a. $\overrightarrow{AB} = \overrightarrow{CB}$ implies $A = C$. A and C must name the same point, the endpoint of the ray.



b. $\overrightarrow{BA} = \overrightarrow{BC}$ indicates only that A and C are on the same ray. It is not necessary that $A = C$.



or

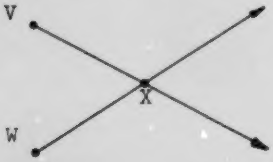


3. a. $\overline{BE}$
   b. B
   c. $\overrightarrow{CA}$
   d. $\overline{CD}$
   e. $\overleftrightarrow{AB}$ or any other notation for the line

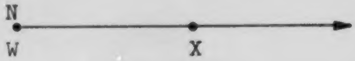
4. a. $\angle PQR$; $\angle Q$; $\angle RQP$
   b. $\overrightarrow{QP}$ and $\overrightarrow{QR}$

5. $\angle CFE$; $\angle CFA$; $\angle BFD$; $\angle DFE$; $\angle DFC$; $\angle CFB$; $\angle BFA$; $\angle BFE$; $\angle DFA$

6. No. There are two possible figures named $\overrightarrow{VX}$ and $\overrightarrow{WX}$:



Case 1



Case 2

In order for $\overrightarrow{VX}$ to be a side of an angle, V must be the vertex, similarly, for $\overrightarrow{WX}$ to be a side, W must be the vertex. This is not true in Case 1. In Case 2, $V = W$ but $\overrightarrow{VX}$ and $\overrightarrow{WX}$ are not two rays, so the definition of an angle is not satisfied.

7. $\angle CFA$; $\angle CFE$; $\angle BFD$

8. a, b, c

9. a, b, d

10. a, c

11. a, b, c, e, f

12. a, c, e

Chapter 12

ADDITION AND SUBTRACTION TECHNIQUES

INTRODUCTION

We have used sets to describe addition and subtraction and to develop its properties. Knowing that $5 + 3$ is the number of members in $A \cup B$, where A is a set of 5 members and B is a disjoint set of 3 members, we may count the members of $A \cup B$ and discover that $5 + 3$ is 8. Knowing that $5 + 3 = 8$, from the definition of subtraction, we can see that $8 - 3 = 5$. This is fine, but it does not really help us much if we want to determine $892 + 367$ or $532 - 278$. To do problems like these quickly and accurately is a goal of real importance. It is a goal whose achievement is made much easier in our decimal system of numeration than in, for instance, the Chinese or Egyptian systems.

This chapter is concerned with explaining the whys and wherefores of so-called "carrying" and "borrowing" in the processes of computing sums and differences. Regrouping is a more accurate term for "carrying" and "borrowing" and will be used throughout this text.

We must recall how our system of numeration with base ten is built. What does the numeral 532 stand for? It stands for $500 + 30 + 2$; or 5 hundreds + 3 tens + 2 ones; or again, since one hundred stands for 10 tens, 532 stands for 5 groups of ten tens + 3 groups of ten + 2 ones. Also if we know that a number has 2 groups of ten tens and 7 groups of ten and 8 ones, we can write a numeral for that number in the form $(2 \times [10 \times 10]) + (7 \times 10) + (8 \times 1)$ or $200 + 70 + 8 = 278$. When we write the numeral in this stretched-out way, we have written it in expanded form.

REGROUPING USED IN ADDITION

Let us assume that we know the addition facts for all the one-digit whole numbers and that we understand our decimal system of numeration. How does this help us? Let's try some examples. Suppose we want the sum of 42 and 37. Since we are adding (4 tens + 2 ones) and

(3 tens + 7 ones) we get (7 tens + 9 ones) which we can write as 79 .

Essentially what we are doing is finding how many groups of tens and how many units we have and then using our system of numeration to write the correct numeral. We may show this in several different forms or algorithms, such as:

(a)
$$\boxed{\begin{array}{l} 3 \text{ tens} + 7 \text{ ones} \\ \underline{4 \text{ tens} + 2 \text{ ones}} \\ 7 \text{ tens} + 9 \text{ ones} = 79 \end{array}}$$

(b)
$$\boxed{\begin{array}{l} 30 + 7 \\ \underline{40 + 2} \\ 70 + 9 = 79 \end{array}}$$

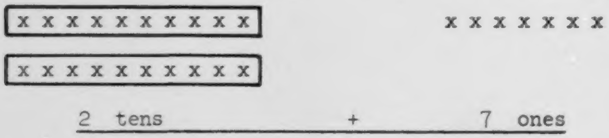
(c)
$$\boxed{\begin{array}{rl} 37 & \\ + \underline{42} & \\ 9 & (7 + 2) \\ \underline{70} & (30 + 40) \\ 79 & \end{array}}$$

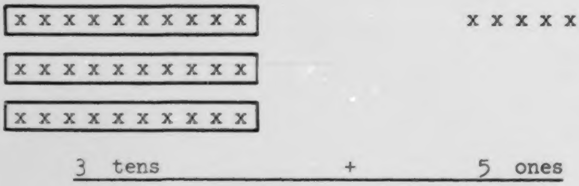
Or we may use an equation form such as

$$\begin{aligned} 37 + 42 &= (30 + 7) + (40 + 2) \\ &= (30 + 40) + (7 + 2) \\ &= 70 + 9 \\ &= 79 \end{aligned}$$

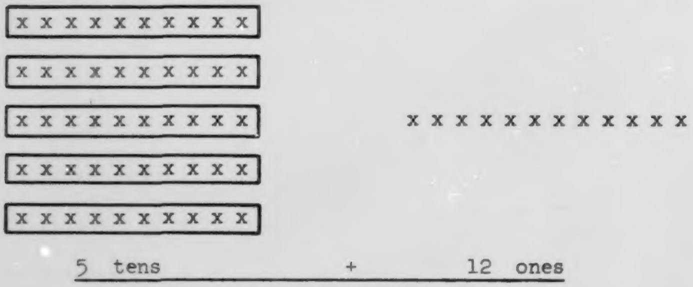
Applying the associative and commutative properties

Let us now add 27 and 35 . This time we have (2 tens + 7 ones) + (3 tens + 5 ones) which may be illustrated:

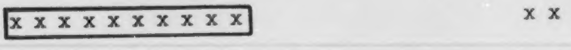

x x x x x x x x x x
x x x x x x x x x x
x x x x x x x
2 tens + 7 ones


x x x x x x x x x x
x x x x x x x x x x
x x x x x x x x x x
x x x x x
3 tens + 5 ones

By putting these groups together we now have:

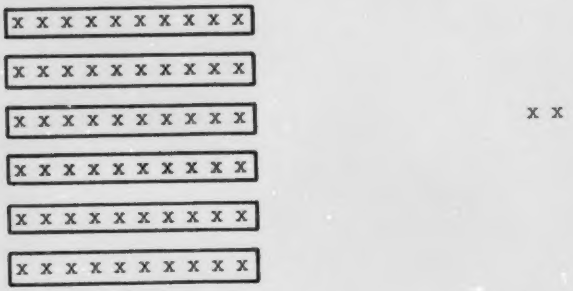

x x x x x x x x x x
x x x x x x x x x x
x x x x x x x x x x
x x x x x x x x x x
x x x x x x x x x x
x x x x x x x x x x x x
5 tens + 12 ones

We now regroup the 12 ones and get another set of 1 ten and 2 ones.



1 ten + 2 ones

We now add (5 tens + 1 ten) + 2 ones.



5 tens + 1 ten + 2 ones

= 6 tens + 2 ones = 62

Or, algorithms such as these may be used:

(a)

2 tens + 7 ones
3 tens + 5 ones
5 tens + 12 ones, or
5 tens + 1 ten + 2 ones, or
6 tens + 2 ones = 62

(b)

20 + 7
30 + 5
50 + 12, or
50 + 10 + 2, or
60 + 2 = 62

(c)

$$\begin{array}{rl} & 27 \\ + & \underline{35} \\ & 12 \quad (7 + 5) \\ & \underline{50} \quad (20 + 30) \\ & 62 \end{array}$$

Using an equation form we may write:

$$\begin{aligned} 27 + 35 &= (20 + 7) + (30 + 5) \\ &= (20 + 30) + (7 + 5) && \text{Applying the associative and commutative properties} \\ &= 50 + 12 \\ &= 50 + (10 + 2) \\ &= (50 + 10) + 2 && \text{Applying the associative property} \\ &= 60 + 2 \\ &= 62 \end{aligned}$$

We may extend these same ideas to the addition of two whole numbers, each greater than 100 . Suppose, for instance, that we were adding 568 and 275 :

(a)

$$\boxed{\begin{array}{lllllll}
5 & \text{hundreds} + & 6 & \text{tens} + & 8 & \text{ones} & \\
2 & \text{hundreds} + & 7 & \text{tens} + & 5 & \text{ones} & \\
\hline
7 & \text{hundreds} + & 13 & \text{tens} + & 13 & \text{ones, or} & \\
7 & \text{hundreds} + & 14 & \text{tens} + & 3 & \text{ones, or} & \\
8 & \text{hundreds} + & 4 & \text{tens} + & 3 & \text{ones} & = 843
\end{array}}$$

or we may write

(b)

$$\boxed{\begin{array}{l}
500 + \ 60 + \ 8 \\
\underline{200 + \ 70 + \ 5} \\
700 + 130 + 13\ \text{, or} \\
700 + 140 + \ 3\ \text{, or} \\
800 + \ 40 + \ 3 = 843
\end{array}}$$

or (c)

$$\boxed{\begin{array}{rl}
568 & \\
+\ \underline{275} & \\
13 & (8 + 5) \\
130 & (60 + 70) \\
\underline{700} & (500 + 200) \\
843 &
\end{array}}$$

Precisely the same process is used in adding three or more numbers. Once again the properties of addition are important. Thus:

$563 + 787 + 1384$ can be thought of as follows:

$$\begin{array}{rcrcr}
563 & = & 500 + 60 + 3 & = & (5 \times 100) + (6 \times 10) + (3 \times 1) \\
787 & = & 700 + 80 + 7 & = & (7 \times 100) + (8 \times 10) + (7 \times 1) \\
\underline{1384} & = & \underline{1000 + 300 + 80 + 4} & = & \underline{(1 \times 1000) + (3 \times 100) + (8 \times 10) + (4 \times 1)} \\
 & & & & (1 \times 1000) + (15 \times 100) + (22 \times 10) + (14 \times 1)
\end{array}$$

and the sum $563 + 787 + 1384$

$$\begin{aligned}
&= (1 \times 1000) + (15 \times 100) + (22 \times 10) + (14 \times 1) \\
&= (1 \times 1000) + [(1 \times 1000) + (5 \times 100)] + [(2 \times 100) + \\
&\qquad (2 \times 10)] + [(1 \times 10) + (4 \times 1)] \\
&= [(1 \times 1000) + (1 \times 1000)] + [(5 \times 100) + (2 \times 100)] + \\
&\qquad [(2 \times 10) + (1 \times 10)] + (4 \times 1) \\
&= (2 \times 1000) + (7 \times 100) + (3 \times 10) + (4 \times 1) \\
&= 2000 + 700 + 30 + 4 \\
&= 2734
\end{aligned}$$

This is usually abbreviated a great deal. But it is important that the underlying pattern be understood and the abbreviations recognized. Thus:

$$\begin{array}{rrrrrrr} & 500 & + & 60 & + & 3 \\ & 700 & + & 80 & + & 7 \\ \hline 1000 + & 300 & + & 80 & + & 4 \\ \hline 1000 + & 1500 & + & 220 & + & 14 \end{array}$$

can be written with partial sums indicated as:

$$\begin{array}{rl} 563 & \\ 787 & \\ 1384 & \\ \hline 14 & \text{sum of ones} \\ 220 & \text{sum of tens} \\ 1500 & \text{sum of hundreds} \\ 1000 & \text{sum of thousands} \\ \hline 2734 & \end{array}$$

and the operation may be still further abbreviated to:

$$\begin{array}{r} ①②① \\ 563 \\ 787 \\ 1384 \\ \hline 2734 \end{array}$$

Finally, by omitting even the "carry over" numerals we get:

$$\begin{array}{r} 563 \\ 787 \\ 1384 \\ \hline 2734 \end{array}$$

PROBLEMS*

1. Find the sum, $38 + 73 + 22$, by an algorithm that shows clearly how the sum is obtained from the addition facts for 0 through 9 only.
2. Show the individual steps required in applying the associative and commutative laws to show that

$$(30 + 7) + (50 + 8) = (30 + 50) + (7 + 8) .$$

A PROPERTY OF SUBTRACTION

Just as we worked the same problem by various methods to get an insight into the addition process, we shall now study the subtraction process by examining various techniques. Let us use a simple example to illustrate the procedures.

Using an equation form for finding the value of the unknown addend n in $n + 23 = 58$ and comparing this with the usual algorithm identifies a property of subtraction that is used extensively in computational work. We write:

$$58 - 23 = (50 + 8) - (20 + 3) .$$

The property of subtraction that deserves our special attention is that which will enable us to express $(50 + 8) - (20 + 3)$ in a useful form.

---

*Solutions for problems in this chapter are on page 211.

The usual procedure for subtracting is by the vertical alignment,

$$\begin{array}{r}58\\ \underline{23}\end{array}\ ,$$

which may be expressed as either of the following:

$$\text{(a)}\quad \boxed{\begin{array}{l}5\ \text{ tens} + 8\ \text{ ones}\\ \underline{2\ \text{ tens} + 3\ \text{ ones}}\\ 3\ \text{ tens} + 5\ \text{ ones} = 35\end{array}} \qquad \text{(b)}\quad \boxed{\begin{array}{l}50 + 8\\ \underline{20 + 3}\\ 30 + 5 = 35\end{array}}$$

In the algorithm (b) above, notice that 3 is subtracted from 8 and 20 is subtracted from 50 to arrive at the tens and ones in the difference. In equation form, this entire process is written:

$$\begin{aligned}58 - 23 = (50 + 8) - (20 + 3) &= (50 - 20) + (8 - 3)\\ &= 30 + 5\\ &= 35\end{aligned}$$

We may state the property, which allows $(50 + 8) - (20 + 3)$ to be reexpressed as $(50 - 20) + (8 - 3)$, more generally in the following way:

IF ONE NUMBER IS $a + b$ AND A
SECOND NUMBER IS $c + d$, AND IF
$a \geq c$ AND $b \geq d$, THEN $(a + b) -$
$(c + d) = (a - c) + (b - d)$

We shall see repeated use of this property, along with regrouping, throughout the rest of this chapter.

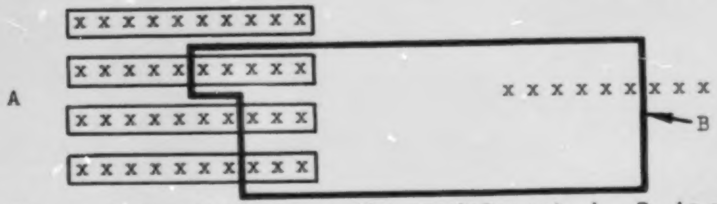
Next, let us interpret subtraction, such as 17 from 49, in terms of set removal. From a set, A, of 49 objects remove a subset, B, of 17 objects, leaving a remainder set, A - B, whose number is to be specified.

We can take for A a collection of 49 x's arranged as follows:

$$A\quad \begin{array}{l}\boxed{x\ x\ x\ x\ x\ x\ x\ x\ x\ x}\\ \boxed{x\ x\ x\ x\ x\ x\ x\ x\ x\ x}\\ \boxed{x\ x\ x\ x\ x\ x\ x\ x\ x\ x}\\ \boxed{x\ x\ x\ x\ x\ x\ x\ x\ x\ x}\end{array} \qquad\qquad x\ x\ x\ x\ x\ x\ x\ x\ x$$

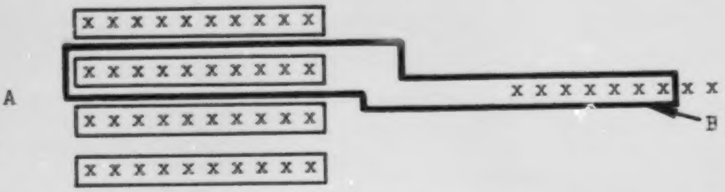
Now we need to pick a subset B of A which contains 17 members. Then the number of members of the remainder set A - B will be 49 - 17 .

There are many ways to choose B . One of them is this:



But when we choose B this way, the remainder set A - B is not easy to count. Some of the original bundles of ten have been broken up, and only pieces of them are in A - B .

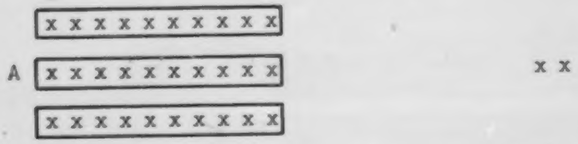
It is much better if we choose B so as to either include all of a bundle of ten or none of it. Here is one way:



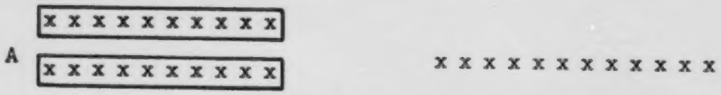
Now it is easy to count the remainder set A - B . It can be done in two steps. Looking at the right hand side above, we see that the number of ones in the remainder set is 9 - 7 = 2 . Looking at the left hand side above, we see that the number of bundles of ten in the remainder set is 4 - 1 = 3 . Therefore the number of members in the remainder set is 32 .

An important thing to notice is that since we dealt only with complete bundles of ten, we could count these using only "small" numbers.

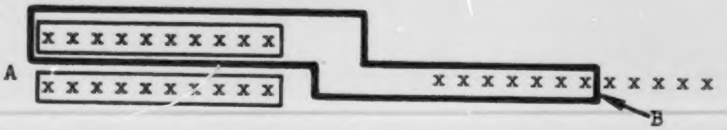
Now, let us examine in the same way another problem: 32 - 17 = n . We can pick A to be a set of 32 x's :



We need to pick a subset B with 17 members, that is, one bundle of ten and seven ones. But A has only two ones, so we will have to use some of the members of A in the bundles of ten. As we saw above, it is best if we use only whole bundles. Therefore, we will take one of the bundles of ten in A, change it to 10 ones, and put it with the 2 ones. Now A looks like this:



Now it is easy to see how we can pick a convenient subset B which has 17 members. Here is one:



It is easy to count the remainder set A - B. The number of ones is 12 - 7 = 5 and the number of tens is 2 - 1 = 1. Therefore 32 - 17 is 1 ten and 5 ones, or 15, and n = 15.

Rather than object representation we may use algorithms such as these to subtract 17 from 32:

(a)
$$\begin{array}{lcl} 3 \text{ tens} + 2 \text{ ones} & = & 2 \text{ tens} + 12 \text{ ones} \\ \underline{1 \text{ ten} + 7 \text{ ones}} & = & \underline{1 \text{ ten} + 7 \text{ ones}} \\ & & 1 \text{ ten} + 5 \text{ ones} = 15 \end{array}$$

or

(b)
$$\begin{array}{lcl} 30 + 2 & = & 20 + 12 \\ \underline{10 + 7} & = & \underline{10 + 7} \\ & & 10 + 5 = 15 \end{array}$$

or we may use an equation form, as

$$\begin{aligned} 32 - 17 &= (30 + 2) - (10 + 7) \\ &= (20 + 12) - (10 + 7) \\ &= (20 - 10) + (12 - 7) \\ &= 10 + 5 \\ &= 15 \end{aligned}$$

Notice that the renaming of (30 + 2) as (20 + 12) involves an application of the associative property of addition, in that

$$(30 + 2) = ([20 + 10] + 2) = (20 + [10 + 2]) = (20 + 12) .$$

We may subtract larger numbers, of course, simply by extending the principles and procedures used with smaller numbers. Consider, for instance, subtracting 276 from 523 .

Since we cannot subtract 6 ones from 3 ones nor 7 tens from 2 tens, renaming is required. In detail, we may write:

$$\begin{aligned}
5 \text{ hundreds} + 2 \text{ tens} + 3 \text{ ones} &= 5 \text{ hundreds} + (1 \text{ ten} + 1 \text{ ten}) + 3 \text{ ones}.\\
&= 5 \text{ hundreds} + (1 \text{ ten} + 10 \text{ ones}) + 3 \text{ ones}.\\
&= 5 \text{ hundreds} + 1 \text{ ten} + 13 \text{ ones}.\\
&= (4 \text{ hundreds} + 1 \text{ hundred}) + 1 \text{ ten} + 13 \text{ ones}.\\
&= (4 \text{ hundreds} + 10 \text{ tens}) + 1 \text{ ten} + 13 \text{ ones}.\\
&= 4 \text{ hundreds} + 11 \text{ tens} + 13 \text{ ones}.
\end{aligned}$$

Ordinarily this procedure is simply indicated by

$$5 \text{ hundreds} + 2 \text{ tens} + 3 \text{ ones} = 4 \text{ hundreds} + 11 \text{ tens} + 13 \text{ ones}.$$

We may now complete the problem 523 - 276 by writing:

$$\begin{array}{rcl}
5 \text{ hundreds} + 2 \text{ tens} + 3 \text{ ones} & = & 4 \text{ hundreds} + 11 \text{ tens} + 13 \text{ ones}\\
\underline{2 \text{ hundreds} + 7 \text{ tens} + 6 \text{ ones}} & = & \underline{2 \text{ hundreds} + 7 \text{ tens} + 6 \text{ ones}}\\
 & & 2 \text{ hundreds} + 4 \text{ tens} + 7 \text{ ones} = 247
\end{array}$$

or we may write

$$\begin{array}{rcl}
500 + 20 + 3 & = & 400 + 110 + 13\\
\underline{200 + 70 + 6} & = & \underline{200 + 70 + 6}\\
 & & 200 + 40 + 7 = 247
\end{array}$$

or we may use an equation form, such as

$$\begin{aligned}
523 - 276 &= (500 + 20 + 3) - (200 + 70 + 6)\\
&= (400 + 110 + 13) - (200 + 70 + 6)\\
&= (400 - 200) + (110 - 70) + (13 - 6)\\
&= 200 + 40 + 7\\
&= 247 .
\end{aligned}$$

We eventually may shorten such algorithms to the form

$$\begin{array}{r}
④⑪⑬\\
5\ 2\ 3\\
-\ \underline{2\ 7\ 6}\\
2\ 4\ 7
\end{array}
\qquad \text{or simply} \qquad
\begin{array}{r}
523\\
-\ \underline{276}\\
247
\end{array}$$

PROBLEMS

3. a. In the property $(a + b) - (c + d) = (a - c) + (b - d)$, why are the conditions $a \geq c$ and $b \geq d$ needed?

   b. Give an illustration of the difficulty encountered if the conditions are not met.

4. a. Represent with an appropriate set, A, and subset, B, the subtraction of 43 and 27.

   b. Show the same subtraction in equation form.

SUMMARY

Techniques of addition and subtraction may be explained in terms of our decimal numeration system, coupled with regrouping and applications of the commutative and associative properties of addition. Subtraction techniques utilize a special property of subtraction; namely,

If a, b, c, and d are whole numbers such that $a \geq c$ and $b \geq d$, then it is true that

$$(a + b) - (c + d) = (a - c) + (b - d) .$$

This special property may be explained in terms of the definition of subtraction in relation to addition, coupled with the commutative and associative properties of addition.

APPLICATIONS TO TEACHING

If young children are to compute with understanding, it is essential that they have an adequate understanding of our numeration system with its base of ten and its principle of place value. They also need to have ample opportunity to manipulate sets of objects as the basis for developing appropriate algorithms.

Algorithms such as these grow readily from manipulations of sets of objects:

1. 42 + 36 = ?

(a) 4 tens + 2 ones	(b) 40 + 2	(c) 42
$\underline{3 \text{ tens} + 6 \text{ ones}}$	$\underline{30 + 6}$	$+ \underline{36}$
7 tens + 8 ones = 78	70 + 8 = 78	8
		$\underline{70}$
		78

2. 69 - 24 = ?

(a) 6 tens + 9 ones	(b) 60 + 9
$\underline{2 \text{ tens} + 4 \text{ ones}}$	$\underline{20 + 4}$
4 tens + 5 ones = 45	40 + 5 = 45

These same algorithms serve young children well when regrouping and renaming are involved:

3. 58 + 17 = ?

(a) 5 tens + 8 ones	(b) 50 + 8	(c) 58
$\underline{1 \text{ ten} + 7 \text{ ones}}$	$\underline{10 + 7}$	$+ \underline{17}$
6 tens + 15 ones, or	60 + 15 , or	15
7 tens + 5 ones = 75	70 + 5 = 75	$\underline{60}$
		75

4. 81 - 35 = ?

(a) 8 tens + 1 one = 7 tens + 11 ones	(b) 80 + 1 = 70 + 11
$\underline{3 \text{ tens} + 5 \text{ ones}} = \underline{3 \text{ tens} + 5 \text{ ones}}$	$30 + 5 = \underline{30 + 5}$
4 tens + 6 ones = 46	40 + 6 = 46

Each child is not expected to be equally at ease with all algorithms. He should be encouraged to work with the form with which he is most comfortable. Eventually he will shorten that algorithm to a more efficient form, but he should not be hurried into doing this. Computing with understanding takes precedence over computing with a highly efficient form in the earlier stages of learning.

QUESTION

"Does the property, $(a + b) - (c + d) = (a - c) + (b - d)$ if $a \geq c$ and $b \geq d$ mean that we cannot perform subtraction for whole numbers if the requirements $a \geq c$ and $b \geq d$ are not met?

To a certain extent, this assumption is correct, but with this assumption, is a distortion in interpretation. If neither of these requirements is met, then it is true that there is no whole number for $(a + b) - (c + d)$. For example, if $a = 2$, $b = 3$, $c = 4$, and $d = 5$, then

$$\begin{aligned}(a + b) - (c + d) &= (2 + 3) - (4 + 5)\\ &= 5 - 9.\end{aligned}$$

Since $5 - 9$ is not a whole number, subtraction cannot be performed for $(2 + 3) - (4 + 5)$ in the set of whole numbers. By the same token, neither can $(20 + 3) - (40 + 5)$ be performed, as can be seen also in the vertical arrangement:

$$\begin{array}{r} 23 \\ -\ 45 \\ \hline \end{array}$$

However, as illustrated in the example $32 - 17$, which is $(30 + 2) - (10 + 7)$, we have $30 > 10$ but $2 < 7$. Still, it is possible to perform this subtraction in the set of whole numbers; we rename $30 + 2$ as $20 + 12$ by regrouping. Then,

$$32 - 17 = (20 + 12) - (10 + 7),$$

and the requirements $a \geq c$ and $b \geq d$ are fulfilled. The only time that such renaming cannot occur to satisfy the requirements is when $(a + b) < (c + d)$; for example, $(20 + 5) - (40 + 3)$ cannot be performed in the set of whole numbers.

VOCABULARY

Addition *
Algorithms *
Associative Property of Addition *
Borrowing
Carrying
Commutative Property of Addition *
Expanded Form *
Regrouping
Subtraction *

EXERCISES - CHAPTER 12

1. For each of these examples, compute using the three addition algorithms just illustrated in the preceding section.

   a. $246 + 139 = ?$

   b. $777 + 964 = ?$

   c. $486 + 766 = ?$

   d. $774 + 926 = ?$

2. For each of these examples, compute using the two subtraction algorithms illustrated in the preceding section.

   a. $764 - 199 = ?$

   b. $402 - 138 = ?$

   c. $710 - 287 = ?$

   d. $800 - 396 = ?$

3. Compute $774 + 926$ using an equation form.

4. Compute $800 - 396$ using an equation form.

SOLUTIONS FOR PROBLEMS

1.

$$\begin{aligned}
38 + 73 + 22 &= 38 + (73 + 22) \\
&= 38 + [(7 \text{ tens} + 3 \text{ ones}) + (2 \text{ tens} + 2 \text{ ones})] \\
&= 38 + [(7 \text{ tens} + 2 \text{ tens}) + (3 \text{ ones} + 2 \text{ ones})] \\
&= 38 + (9 \text{ tens} + 5 \text{ ones}) \\
&= 38 + 95 \\
&= (3 \text{ tens} + 8 \text{ ones}) + (9 \text{ tens} + 5 \text{ ones}) \\
&= (3 \text{ tens} + 9 \text{ tens}) + (8 \text{ ones} + 5 \text{ ones}) \\
&= 12 \text{ tens} + 13 \text{ ones} \\
&= (1 \text{ hundred} + 2 \text{ tens}) + (1 \text{ ten} + 3 \text{ ones}) \\
&= 1 \text{ hundred} + (2 \text{ tens} + 1 \text{ ten}) + 3 \text{ ones} \\
&= 1 \text{ hundred} + 3 \text{ tens} + 3 \text{ ones} \\
&= 133
\end{aligned}$$

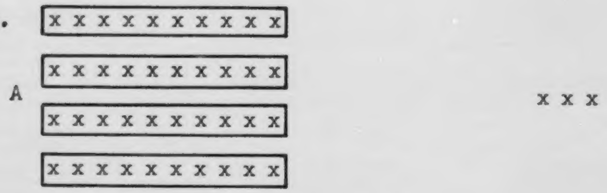
2.

$$\begin{aligned}
(30 + 7) + (50 + 8) &= ([30 + 7] + 50) + 8 && \textbf{associative property} \\
&= (30 + [7 + 50]) + 8 && \textbf{associative property} \\
&= (30 + [50 + 7]) + 8 && \textbf{commutative property} \\
&= ([30 + 50] + 7) + 8 && \textbf{associative property} \\
&= (30 + 50) + (7 + 8) && \textbf{associative property}
\end{aligned}$$

3. a. In order for $a - c$ and $b - d$ to have meaning, it is necessary that $a \geq c$ and $b \geq d$. These conditions also assure that $a + b \geq c + d$ which makes $(a + b) - (c + d)$ meaningful.

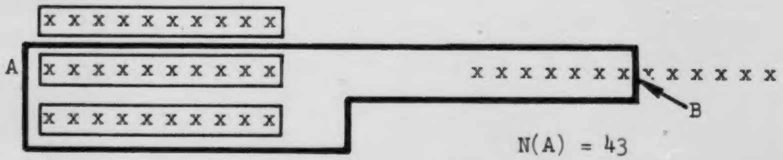
b. For example, let $a = 7$, $b = 5$, $c = 8$, $d = 2$, so that $a \geq c$ is not true. Then $(a + b) - (c + d) = (7 + 5) - (8 + 2) = 12 - 10 = 2$, and $(a - c) + (b - d) = (7 - 8) + (5 - 2) = (7 - 8) + 3 = ?$ $7 - 8$ is not a whole number, so the property is undefined. If neither condition had been true, $(a + b) - (c + d)$ would not have been defined.

4. a.



$N(A) = 43$

or, regrouped,



$N(A) = 43$

$N(B) = 27$

$N(A \sim B) = 43 - 27 = 16$

b.
$$\begin{aligned} 43 - 27 &= (40 + 3) - (20 + 7) \\ &= (30 + 13) - (20 + 7) \\ &= (30 - 20) + (13 - 7) \\ &= 10 + 6 \\ &= 16 \end{aligned}$$

Chapter 13

INTRODUCING RATIONAL NUMBERS

INTRODUCTION

All our work with numbers up to this point has been with the set of whole numbers; we have pretended that they are the only numbers that exist and we have seen how they and their operations behave. Our number lines have been marked only at the points which correspond to whole numbers, leaving gaps containing many points that are not named. Using only whole numbers it is clear that many division problems cannot be worked (for example $3 \div 4$); that is, the set of whole numbers is not closed under the operation of division.

Now the problem of assigning numbers to "parts of wholes", the problem of naming points between those named by whole numbers on the number line, and the problem of lack of closure under division of whole numbers are three problems that convince us of the need to extend our number system to include more than the whole numbers. In the historical development of numbers the problem of measurement (which will be considered in Chapter 16) was probably a significant motivation in forcing the extension of number systems to more sophistication than merely counting and numbering.

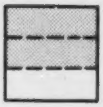
REGIONS AS MODELS FOR RATIONAL NUMBERS

In our extension of the number system to include what we will call rational numbers (but which are frequently called "fractional numbers") we will proceed much as we did with the whole numbers. That is, we will start with physical models for such numbers and from these develop some concepts about them.

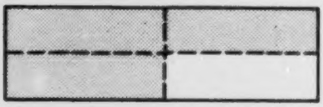
In setting up physical models for rational numbers we usually begin by designating some "basic unit", for example, a rectangular region, a circular region, a segment, or a collection of things. This basic unit is then partitioned into a certain number of congruent parts. These parts, compared to the unit, give us the basis for a model for rational numbers.

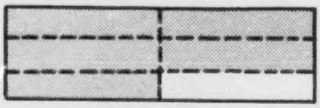
For example, let us identify as our basic unit a square region and suppose this is partitioned into two congruent parts as shown in Figure (a). We want to associate a number with the shaded part of the square region. Not only do we want a number, we want a name for this number, a numeral which will remind us of the two congruent parts we have, of which one is shaded. The numeral is the obvious one, $\frac{1}{2}$, read "one-half". If our unit is partitioned into three congruent parts and if two of them are shaded, as in Figure (b), the numeral $\frac{2}{3}$ reminds us that we are associating a number with two of three congruent parts of a unit. Observe that our numeral still uses notions expressible by whole numbers; that is, a basic unit is partitioned into three congruent parts with two of these considered.

(a)


(b)

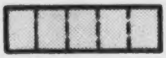
In the figures below, a rectangular region serves as the unit.


(c)

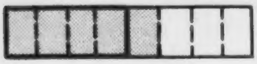

(d)

The numeral $\frac{3}{4}$ expresses the situation pictured in Figure (c), namely the unit region partitioned into four congruent regions, of which three are shaded. And, of course, the numeral $\frac{5}{6}$ expresses the situation represented by Figure (d), the basic unit partitioned into six congruent regions, of which five regions are shaded.

More complicated situations are represented in the next drawings. In each case the basic unit is the rectangular region heavily outlined by solid lines. In some of these, the shaded region designates a region the same as or more than the basic region, hence numbers equal to or greater than one. Thus Figure (e) shows the basic unit partitioned into five parts, all of which are shaded. The numeral $\frac{5}{5}$ describes this model.

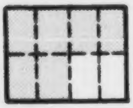


(e) Physical model for $\frac{5}{5}$

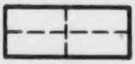


Unit

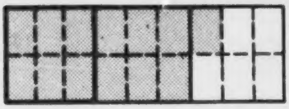
(f) Physical model for $\frac{5}{4}$



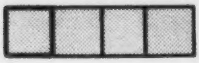
(g) $\frac{6}{8}$



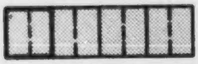
(h) $\frac{0}{4}$



(i) $\frac{13}{6}$

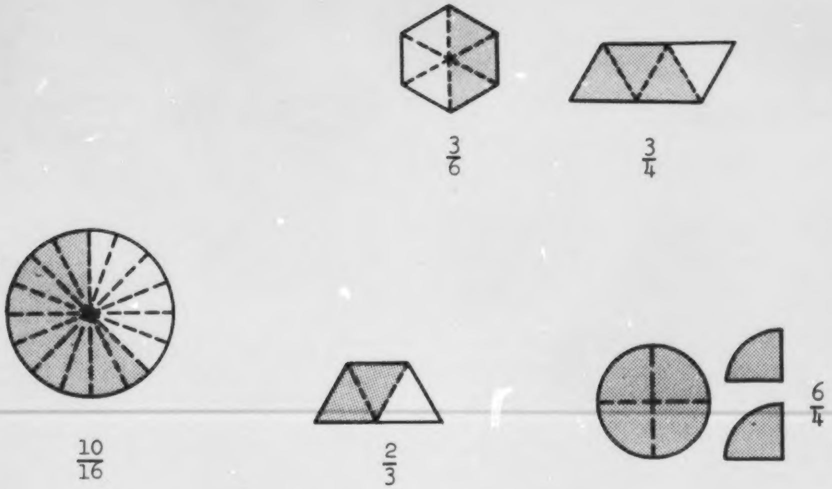


(j) $\frac{4}{1}$



(k) $\frac{8}{2}$

In Figure (f), the unit region is partitioned into four congruent regions, and five such regions are shaded; the numeral $\frac{5}{4}$ describes this model. Examine the other situations illustrated and verify that in each case the region shaded is indeed a model for the rational number named under it.



Models using regions of various shapes

Regions of other shapes can also be used as models for rational numbers. Some such regions, with associated numerals, are pictured above. In each case, you can verify that the model involves identification of a unit region, partitioning of this region into congruent regions, and consideration of a certain number of these congruent regions.

For the sake of simplicity, we have used as models only plane regions. Frequently, we use solid regions, also, as models for rational numbers. The interpretation given to such models is but an extension of that used with plane regions. In this chapter we shall only use plane regions as unit regions.

PROBLEMS*

1. Draw models for:

a. $\frac{2}{3}$

b. $\frac{4}{6}$

c. $\frac{3}{2}$

d. $\frac{12}{5}$

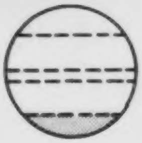
e. $\frac{7}{7}$

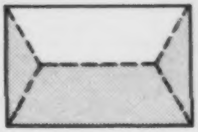
f. $\frac{0}{6}$

---

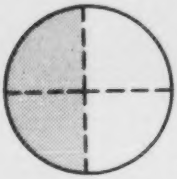
*Solutions for the problems in this chapter are on page 237.

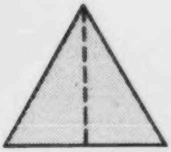
2. Why are the following pictures not good models for rational numbers?

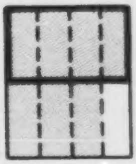


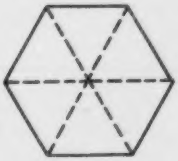


3. What numbers do the shaded portions of the following models illustrate?


(a)

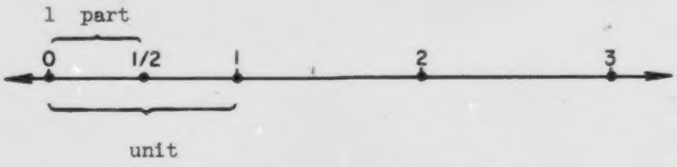

(b)


(c)

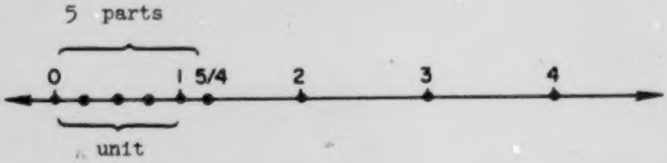

(d)

NUMBER LINE MODELS FOR RATIONAL NUMBERS

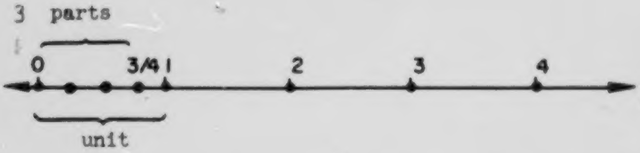
Another standard physical model for the idea of a rational number uses the number line. The way we locate new points on the number line parallels the procedure we followed with regions. After we mark off a unit segment and partition it into congruent segments, we then count these parts. Thus, in order to locate the point corresponding to $\frac{1}{2}$, we mark off the unit segment into 2 congruent parts and count off 1 of them. This point corresponds to $\frac{1}{2}$.

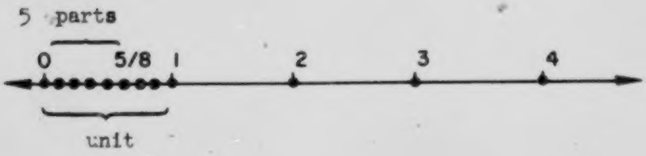


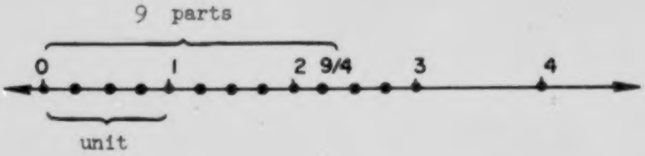
In like manner, to locate $\frac{5}{4}$, we partition a unit interval into 4 congruent parts and count off 5 of these parts. We have now located the point which we associate with $\frac{5}{4}$.


5 parts
0
1
5/4
2
3
4
unit

Once we have this method in mind, we see that we can associate a point on the number line with all symbols such as $\frac{3}{4}$, $\frac{5}{8}$, $\frac{9}{4}$, etc., as illustrated below.


3 parts
0
3/4
1
2
3
4
unit


5 parts
0
5/8
1
2
3
4
unit


9 parts
0
1
2
9/4
3
4
unit

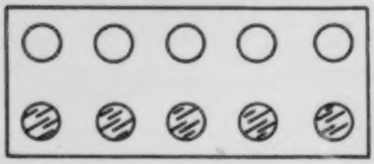
PROBLEM

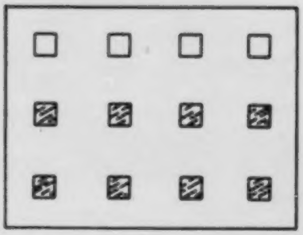
4. Locate the point associated with each of the following on a separate number line.

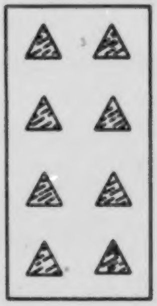
a.	$\frac{0}{1}$	d.	$\frac{5}{5}$
b.	$\frac{3}{4}$	e.	$\frac{7}{4}$
c.	$\frac{3}{5}$	f.	$\frac{8}{8}$

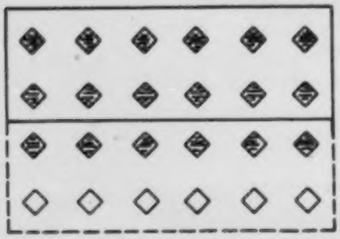
ARRAY MODELS FOR RATIONAL NUMBERS

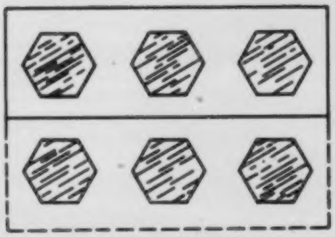
Sets of objects arranged in arrays may serve as models for rational numbers, as in the illustrations below. In each figure the unit set or array is bounded by solid lines.


(a) A model for $\frac{1}{2}$

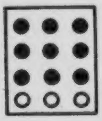

(b) A model for $\frac{2}{3}$

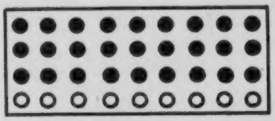

(c) A model for $\frac{4}{4}$


(d) A model for $\frac{3}{2}$

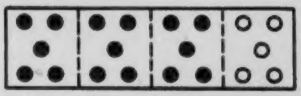

(e) A model for $\frac{2}{1}$

In Figure (a), for instance, one of the two rows of the unit array is shaded. With this model we may associate the rational number $\frac{1}{2}$. In Figure (c), four of the four rows of the unit array are shaded, and with this model we may associate the rational number $\frac{4}{4}$. There are two unit arrays in Figure (d) with two rows in each array. Three of the rows are shaded, and with this model we may associate the rational number $\frac{3}{2}$. Notice that in each instance the rational number associated with a particular model is independent of the number of elements in each row of the array. For example: we would associate the same rational number, $\frac{3}{4}$, with either of the arrays below.





Notice that we also may associate the rational number $\frac{3}{4}$ with a representation that is not an array, such as:



in which a unit set is partitioned into four equivalent subsets, three of which are to be considered.

PROBLEM

5. Show an array as a model for each of these.

a. $\frac{5}{6}$ b. $\frac{3}{8}$ c. $\frac{7}{7}$ d. $\frac{4}{3}$ e. $\frac{7}{4}$ f. $\frac{5}{2}$

SOME VOCABULARY AND OTHER CONSIDERATIONS

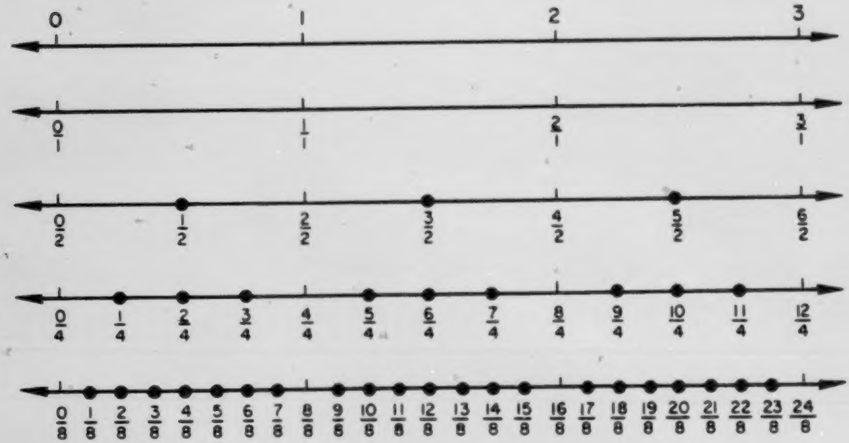
The numbers for which our regions, segments, and arrays are models are called rational numbers. The particular numeral form in which these numbers often are expressed is called a fraction. Many different fractions designate the same rational number. We have here again the distinction between a number and names (numerals) for that number.

In this chapter we are concerned with those rational numbers that can be named by a fraction of the form $\frac{a}{b}$ where a represents a whole number and b represents a counting number (i.e., a whole number other than zero). In effect, this definition restricts us to a consideration of the nonnegative rational numbers. The complete set of rational numbers consists of those numbers of the specified form, $\frac{a}{b}$, and their opposites or negatives.

Referring to our models we see that b , the denominator, always is the number of congruent parts or equivalent subsets into which a unit has been partitioned, while a , the numerator, is the number of these congruent parts or equivalent subsets that are being used. One of several reasons why the denominator is never zero is that it would be nonsense to speak of a unit as being divided into zero parts; it surely cannot be partitioned into fewer than one part.

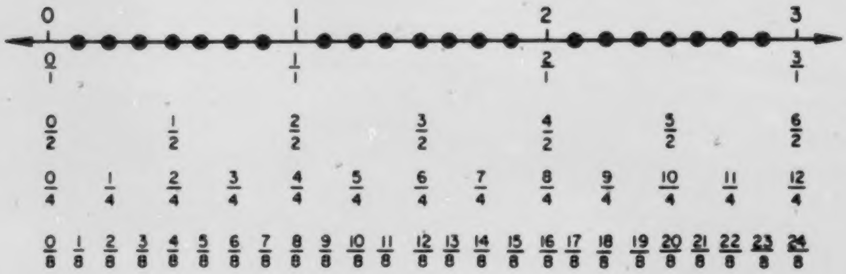
EQUIVALENT FRACTIONS

The following figure shows several number lines: one on which we have located points corresponding to 0 , 1 , 2 , 3 , etc.; one on which we have located points corresponding to $\frac{0}{1}$, $\frac{1}{1}$, $\frac{2}{1}$, $\frac{3}{1}$, etc.; one on which we have located points corresponding to $\frac{0}{2}$, $\frac{1}{2}$, $\frac{2}{2}$, $\frac{3}{2}$, etc.; one on which we have located points corresponding to $\frac{0}{4}$, $\frac{1}{4}$, $\frac{2}{4}$, $\frac{3}{4}$, $\frac{4}{4}$, $\frac{5}{4}$, etc.; and one on which we have located points corresponding to $\frac{0}{8}$, $\frac{1}{8}$, $\frac{2}{8}$, $\frac{3}{8}$, $\frac{4}{8}$, etc.


0 1 2 3
0/1 1/1 2/1 3/1
0/2 1/2 2/2 3/2 4/2 5/2 6/2
0/4 1/4 2/4 3/4 4/4 5/4 6/4 7/4 8/4 9/4 10/4 11/4 12/4
0/8 1/8 2/8 3/8 4/8 5/8 6/8 7/8 8/8 9/8 10/8 11/8 12/8 13/8 14/8 15/8 16/8 17/8 18/8 19/8 20/8 21/8 22/8 23/8 24/8

As we look at these number lines, we see that it seems very natural to think of $\frac{0}{2}$, for example, as being associated with the zero point. For we are really, so to speak, counting off 0 segments. Similarly, it seems natural to locate $\frac{0}{1}$, $\frac{0}{4}$ and $\frac{0}{8}$ as indicated.

Now let us put the five number lines together, as shown in the figure below. In other words let us carry out on a single line the steps

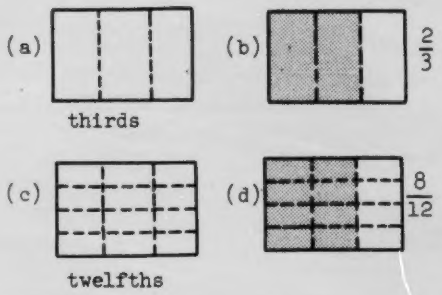


for locating in turn points corresponding to the rational numbers with denominator 1, with denominator 2, with denominator 4 and with denominator 8. When we do this we see, among other things, that $\frac{1}{2}$, $\frac{2}{4}$, and $\frac{4}{8}$ all correspond to the same point on the number line, or, in other words, are all names (numerals) for the same rational number. We see also that $\frac{0}{1}$, $\frac{1}{1}$, $\frac{2}{1}$, and so on, name the points we have formerly named with whole numbers. Furthermore we see that fractions such as $\frac{2}{2}$, $\frac{4}{4}$, $\frac{4}{2}$, $\frac{8}{4}$, and the like also name points that have formerly been named by whole numbers. Fractions which name the same point on the number line, and which therefore name the same rational number, are called equivalent fractions. Notice that corresponding to each whole number there is a set of equivalent fractions. Consequently, there is a one-to-one correspondence between the set of whole numbers and a particular subset of the set of rational numbers. Furthermore, it can be shown that a one-to-one correspondence may be established between the set of whole numbers and the entire set of rationals.

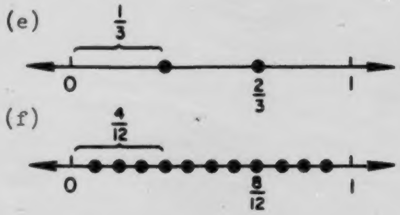
EQUIVALENT FRACTIONS IN "HIGHER TERMS"

Recognizing the same rational number under a variety of disguises (names) and being able to change the names of numbers without changing the numbers are great conveniences in operating efficiently with rational numbers. Such an addition problem as $\frac{1}{4} + \frac{2}{3}$ is certainly worked out most efficiently by considering the equivalent problem $\frac{3}{12} + \frac{8}{12}$, equivalent because $\frac{1}{4}$ names the same number as $\frac{3}{12}$ and $\frac{2}{3}$ names the same number as $\frac{8}{12}$.

The figures illustrate a way of using our unit region model to show that $\frac{2}{3}$ and $\frac{8}{12}$ are equivalent fractions, that is, that $\frac{2}{3}$ and $\frac{8}{12}$ name the same number. First we select a unit region and partition it into three congruent regions by vertical lines as shown in Figure (a). Figure (b) shows the shading of two of these regions to represent $\frac{2}{3}$. If we return now to our unit region and partition each of the former three congruent parts by horizontal lines into four congruent parts, we have the unit partitioned into $3 \times 4 = 12$ congruent parts, as shown in Figure (c) If the unit partitioned in this way is now superimposed on the model for $\frac{2}{3}$, we get the model shown in Figure (d), which shows that each of the two shaded regions in the model for $\frac{2}{3}$ is partitioned into four regions, giving $2 \times 4 = 8$ smaller congruent regions shaded. Hence the model showing 8 of 12 congruent parts represents the same number as the model showing 2 of 3 congruent parts.



Model showing $\frac{2}{3} = \frac{2 \times 4}{3 \times 4} = \frac{8}{12}$.



Number line model showing that $\frac{2}{3} = \frac{2 \times 4}{3 \times 4} = \frac{8}{12}$.

The number lines in Figures (e) and (f) demonstrate this same equivalence. In Figure (e), $\frac{2}{3}$ is shown by partitioning the unit segment into 3 congruent parts and using two of these to mark a point. If each of the 3 congruent parts of the unit is now partitioned into 4 congruent parts, the unit segment then contains $3 \times 4 = 12$ parts while the 2 original parts used to mark $\frac{2}{3}$ now contain $2 \times 4 = 8$ congruent parts, as shown in Figure (f). Hence, the same point is named by $\frac{8}{12}$ as was formerly named by $\frac{2}{3}$.

To put this in more general terms, consider the fraction $\frac{a}{b}$ where b represents the number of parts a unit has been partitioned into and a the number of these parts marked in the model. If each of the b parts is further partitioned into k congruent parts, the unit then contains $b \times k$ congruent parts. At the same time, each of the a parts is further partitioned into k parts so that there will be $a \times k$ smaller congruent parts marked in the model. Hence, $\frac{a \times k}{b \times k}$ represents the same number as $\frac{a}{b}$ formerly did. Symbolically:

$$\frac{a}{b} = \frac{a \times k}{b \times k}$$

where k represents any counting number. Hence, for instance,

$$\frac{3}{4} = \frac{3 \times 2}{4 \times 2} = \frac{6}{8}\text{, or } \frac{3}{4} = \frac{3 \times 3}{4 \times 3} = \frac{9}{12}\text{, or } \frac{3}{4} = \frac{3 \times 4}{4 \times 4} = \frac{12}{16}\text{, etc.}$$

Our knowledge of multiples of numbers can be used to good advantage when each of two fractions such as $\frac{5}{6}$ and $\frac{3}{4}$ is to be changed to "higher terms" so that each fraction has the same denominator.

The set of multiples of 6 is $\{6, 12, 18, 24, 30, 36, \ldots\}$.

The set of multiples of 4 is $\{4, 8, 12, 16, 20, 24, \ldots\}$.

The intersection of these two sets is $\{12, 24, 36, 48, \ldots\}$ and any member of this intersection can serve as the "common denominator" for the new fractions. The least common denominator would be 12, of course, so that

$$\frac{5}{6} = \frac{5 \times 2}{6 \times 2} = \frac{10}{12} \quad\text{and}\quad \frac{3}{4} = \frac{3 \times 3}{4 \times 3} = \frac{9}{12},$$

or alternately,

$$\frac{5}{6} = \frac{5 \times 4}{6 \times 4} = \frac{20}{24} \quad\text{and}\quad \frac{3}{4} = \frac{3 \times 6}{4 \times 6} = \frac{18}{24},$$

and so on.

PROBLEMS

6. Draw both a unit region model and a number line model to illustrate that $\frac{2}{3} = \frac{4}{6}$.

7. Supply the missing numbers in each of the following.

   a. $\frac{3}{5} = \frac{3 \times }{5 \times } = \frac{24}{40}$ b. $\frac{7}{8} = \frac{}{32}$ c. $\frac{}{12} = \frac{14}{24}$

8. Specify the "k" used in each case to change the first fraction to the second.

   a. $\frac{7}{13} = \frac{7 \times k}{13 \times k} = \frac{28}{42}$; k = __

   b. $\frac{14}{16} = \frac{42}{48}$; k = __

   c. $\frac{3}{7} = \frac{63}{147}$; k = __

EQUIVALENT FRACTIONS IN "LOWER TERMS"

Expressing a fraction in "lower terms" (often called "reducing" fractions) is simply reversing, or undoing, the process used to express fractions in "higher terms". For example, $\frac{2}{3} = \frac{2 \times 10}{3 \times 10} = \frac{20}{30}$ and, undoing this process, $\frac{20}{30} = \frac{20 \div 10}{30 \div 10} = \frac{2}{3}$. Similarly, $\frac{10}{4} = \frac{10 \div 2}{4 \div 2} = \frac{5}{2}$, $\frac{12}{18} = \frac{12 \div 3}{18 \div 3} = \frac{4}{6}$, $\frac{147}{3} = \frac{147 \div 3}{3 \div 3} = \frac{49}{1}$ and so on. In general:

IF A COUNTING NUMBER, k, IS A FACTOR OF BOTH a AND b, THEN $\frac{a}{b} = \frac{a \div k}{b \div k}$.

In this case we say that the fraction $\frac{a}{b}$ has been changed to "lower terms". It should be noted that while it is always possible to change a fraction to an equivalent one in "higher terms" with denominator any desired multiple of the original denominator, it is not always possible to rename ("reduce") a fraction using a specified divisor (factor), since one cannot always divide a counting number by a counting number. For example, $\frac{4}{6}$ can be renamed using 2 as a divisor, but not by using 3, while $\frac{3}{5}$ cannot be changed to any "lower terms". We sometimes say that a fraction which cannot be changed to any "lower terms, such as $\frac{1}{3}$, $\frac{4}{7}$, etc., is in simplest form or lowest terms.

Putting fractions in lowest terms or simplest form is a convenient skill, but its importance has been overrated. The superstition that fractions must always, ultimately, be written in this form has no mathematical basis, for only different names for the same number are at issue. It is often convenient for purposes of further computation or to make explicit a particular interpretation to leave results in other than simplest form. However, where simplest form is desired we can proceed by repeated division in both numerator and denominator, or we can use the greatest common factor of both numerator and denominator as the k by which both should be divided. The examples displayed below should be sufficient to illustrate both procedures for writing a fraction in simplest form.

(a) $\frac{12}{20} = \frac{12 \div 2}{20 \div 2} = \frac{6}{10} = \frac{6 \div 2}{10 \div 2} = \frac{3}{5}$

$12 = (2 \times 2) \times 3$

$20 = (2 \times 2) \times 5$

So the greatest common factor of 12 and 20 is $2 \times 2 = 4$, and $\frac{12}{20} = \frac{12 \div 4}{20 \div 4} = \frac{3}{5}$.

(b) $\frac{104}{260} = \frac{104 \div 2}{260 \div 2} = \frac{52}{130} = \frac{52 \div 2}{130 \div 2} = \frac{26}{65} = \frac{26 \div 13}{65 \div 13} = \frac{2}{5}$

$$\begin{array}{ll} 2 \;\underline{|104} & \textcircled{2}\;\underline{|260} \\ \textcircled{2}\;\underline{|52} & \textcircled{2}\;\underline{|130} \\ \textcircled{2}\;\underline{|26} & 5\;\underline{|65} \\ \quad\textcircled{13} & \quad\textcircled{13} \end{array}$$

So the greatest common factor is $2 \times 2 \times 13 = 52$, and

$\frac{104}{260} = \frac{104 \div 52}{260 \div 52} = \frac{2}{5}$.

Observe that for a fraction such as $\frac{5}{9}$ the greatest common factor of 5 and 9 is one, and consequently the fraction already is in its lowest terms. It is true that $\frac{5}{9} = \frac{5 \div 1}{9 \div 1} = \frac{5}{9}$, but there is no need to perform such a division.

PROBLEMS

9. For each of the following, give one equivalent fraction in "higher terms" and give three equivalent fractions in "lower terms", including one in lowest terms.

   a. $\frac{24}{36}$ b. $\frac{30}{60}$

10. Why would it not make sense to speak of a fraction raised to "highest terms"?

11. For each of the following, specify the greatest common factor, say f, of the numerator and denominator and use f to write the fraction in simplest form.

    a. $\frac{30}{45}$ f = _____ $\frac{30}{45} =$

    b. $\frac{24}{36}$ f = _____ $\frac{24}{36} =$

    c. $\frac{39}{52}$ f = _____ $\frac{39}{52} =$

EQUALITY AND ORDER AMONG RATIONAL NUMBERS

First let us recall the three possible relations that may exist between two whole numbers, m and n. One and only one of these three things is true:

$m = n$ (m is equal to n)
$m > n$ (m is greater than n)
$m < n$ (m is less than n)

A similar statement can be made about two rational numbers, $\frac{a}{b}$ and $\frac{c}{d}$:

$\frac{a}{b} = \frac{c}{d}$ ($\frac{a}{b}$ is equal to $\frac{c}{d}$)
$\frac{a}{b} > \frac{c}{d}$ ($\frac{a}{b}$ is greater than $\frac{c}{d}$)
$\frac{a}{b} < \frac{c}{d}$ ($\frac{a}{b}$ is less than $\frac{c}{d}$)

Let us consider these three specific examples:

1. $\frac{6}{8}, \frac{9}{12}$ 2. $\frac{7}{8}, \frac{5}{6}$ 3. $\frac{5}{8}, \frac{4}{6}$

How may we compare the rational numbers in each example to determine whether the first number of each pair is equal to, or greater than, or less than the second number of each pair? Of the several approaches that might be taken, we shall illustrate the one in which each pair of fractions is expressed in terms of equivalent fractions whose denominators are the same. In particular, the common denominator will be the least common denominator. Thus:

1. To compare $\frac{6}{8}$ and $\frac{9}{12}$: since $\frac{6}{8} = \frac{18}{24}$, $\frac{9}{12} = \frac{18}{24}$, and $\frac{18}{24} = \frac{18}{24}$ it must be true that $\frac{6}{8} = \frac{9}{12}$

2. To compare $\frac{7}{8}$ and $\frac{5}{6}$: since $\frac{7}{8} = \frac{21}{24}$, $\frac{5}{6} = \frac{20}{24}$, and $\frac{21}{24} > \frac{20}{24}$, it must be true that $\frac{7}{8} > \frac{5}{6}$.

3. To compare $\frac{5}{8}$ and $\frac{4}{6}$: since $\frac{5}{8} = \frac{15}{24}$, $\frac{4}{6} = \frac{16}{24}$, and $\frac{15}{24} < \frac{16}{24}$, it must be true that $\frac{5}{8} < \frac{2}{3}$.

Now let us summarize each of these three comparisons and also make a significant observation in each instance:

1. $\frac{6}{8} = \frac{9}{12}$. It also is true that $6 \times 12 = 8 \times 9$.

2. $\frac{7}{8} > \frac{5}{6}$. It also is true that $7 \times 6 > 8 \times 5$.

3. $\frac{5}{8} < \frac{2}{3}$. It also is true that $5 \times 3 < 8 \times 2$.

It is extremely dangerous to generalize on the basis of isolated examples! However, the preceding examples do illustrate an important set of relations that can be demonstrated to be true for all nonnegative rational numbers $\frac{a}{b}$ and $\frac{c}{d}$:

$\frac{a}{b} = \frac{c}{d}$ if and only if $a \times d = b \times c$.

$\frac{a}{b} > \frac{c}{d}$ if and only if $a \times d > b \times c$.

$\frac{a}{b} < \frac{c}{d}$ if and only if $a \times d < b \times c$.

Thus, we have a very simple and convenient way for determining whether or not two rational numbers are equal and, if not equal, a very simple and convenient way for ordering them.

PROBLEM

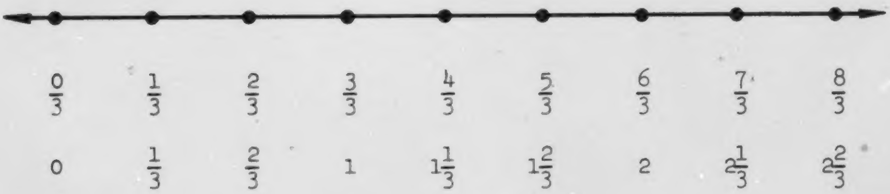
12. Make each of the following statements true by writing $=$ or $>$ or $<$ in the ring.

a. $\frac{6}{14} \bigcirc \frac{7}{16}$ b. $\frac{6}{8} \bigcirc \frac{9}{12}$ c. $\frac{30}{63} \bigcirc \frac{15}{28}$

d. $\frac{3}{4} \bigcirc \frac{36}{52}$ e. $\frac{9}{20} \bigcirc \frac{45}{100}$ f. $\frac{143}{13} \bigcirc \frac{1043}{103}$

RATIONAL NUMBERS IN MIXED FORM

Each of us is familiar with the fact that a rational number whose name is $\frac{3}{2}$, for example, also may be named in the "mixed form" sometimes called a "mixed numeral", $1\frac{1}{2}$. (We do not say "mixed number" because $1\frac{1}{2}$ is a numeral, that is a name for a number, and not a numeral.) Let us use the number line to examine briefly some of the assumptions underlying our use of the familiar mixed form for naming certain rational numbers.



Consider, for instance, the use of $\frac{5}{3}$ and $1\frac{2}{3}$ to name the same rational number. We often state that $\frac{5}{3} = 1\frac{2}{3}$. Behind this statement there is the assumption, among others, that rational numbers can be added: $\frac{5}{3} = \frac{3}{3} + \frac{2}{3} = 1 + \frac{2}{3} = 1\frac{2}{3}$.

Or, consider the statement that $\frac{7}{3} = 2\frac{1}{3}$. Here again we see that the ability to add rational numbers is one of the things underlying our interpretation of $2\frac{1}{3}$, since: $\frac{7}{3} = \frac{6}{3} + \frac{1}{3} = 2 + \frac{1}{3} = 2\frac{1}{3}$.

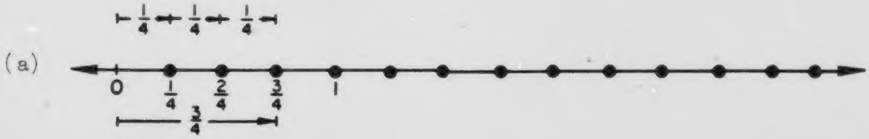
It is beyond the scope of this chapter to give any systematic consideration to the addition of rational numbers. However, we did wish to point out that this operation is implicit in an interpretation of the mixed form for a rational number.

Another important implicit assumption is considered in the following section.

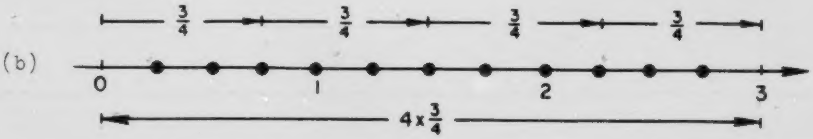
RATIONAL NUMBERS AND DIVISION

Thus far rational numbers have been interpreted in terms of several models: unit regions partitioned into congruent regions, unit sets or arrays partitioned into equivalent subsets, and unit segments partitioned into congruent segments. We shall now look more closely at the interpretation of rational numbers on the number line.

For an example we shall consider $\frac{3}{4}$. We partition the unit segment into four congruent subsegments and count three of them. Each interval in the partition represents $\frac{1}{4}$, therefore three-fourths is the union of three of these subsegments. Numerically this implies that $\frac{3}{4}$ is defined as $3 \times \frac{1}{4}$.



Similarly, the union of four of these segments abutted end-to-end represents $4 \times \frac{3}{4}$ or 3, as shown in (b).



This is consistent with the above definition and the associative property of multiplication for the product:

$$4 \times \frac{3}{4} = 4 \times (3 \times \frac{1}{4}) = (4 \times 3) \times \frac{1}{4} = 12 \times \frac{1}{4} = \frac{12}{4} = 3 .$$

The following equality derived from the preceding work is of particular interest:

$$4 \times \frac{3}{4} = 3$$

It demonstrates that there is a number of the form $\frac{a}{b}$ that satisfies the equation

$$4 \times \frac{a}{b} = 3 ,$$

namely, $n \rightarrow \frac{a}{b}$. Associated with this equation is the quotient $\frac{a}{b} = 3 \div 4$. This had no meaning in the set of whole numbers, but we see now that the set of rational numbers provides the number $\frac{3}{4}$ as equal to $3 \div 4$.

Recall the use of the number line in illustrating division, say of $6 \div 3$. A 6 unit segment is partitioned into 3 congruent subsegments. Each subsegment is congruent to the segment from 0 to 2 , and thus, $6 \div 3 = 2$. A similar partitioning of a 3 unit segment into 4 congruent subsegments can be associated with $3 \div 4$. As Figure (b) above shows, each subsegment is congruent to the segment from 0 to $\frac{3}{4}$, thus justifying further that $3 \div 4 = \frac{3}{4}$.

This is but one illustration of an important relation between rational numbers and division. In general, it is true that

$$a \div b = \frac{a}{b}$$

where a is any whole number, b is any counting number, and their quotient is the rational number $\frac{a}{b}$. Thus, for every whole number a and for every counting number b there is a rational number $\frac{a}{b}$ such that

$$b \times \frac{a}{b} = a .$$

PROBLEM

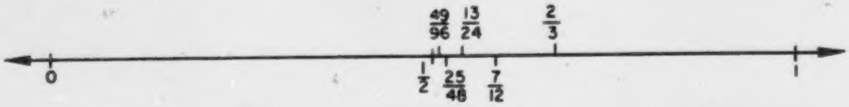
13. a. Find n if $3 \times n = 5$.

    b. Show the division on the number line.

A NEW PROPERTY OF NUMBERS

Rational numbers are different in many ways from whole numbers. One such difference is apparent if we recall that for any whole number one can always say what the "next" whole number is and then ask, in a similar vein, what the "next" rational number is after any given rational number. For example, 4 is the next whole number after 3 , 1069 is the next whole number after 1068 , and so on. What is the next rational number after $\frac{1}{2}$? If $\frac{2}{3}$ is suggested as the next one, we can observe that $\frac{1}{2} = \frac{6}{12}$ and $\frac{2}{3} = \frac{8}{12}$, so $\frac{7}{12}$ is surely between $\frac{1}{2}$ and $\frac{2}{3}$. Hence, $\frac{7}{12}$ has a better claim to being next to $\frac{1}{2}$ than does $\frac{2}{3}$. If it is then suggested that $\frac{7}{12}$ be regarded as the next number after $\frac{1}{2}$, we can observe that $\frac{1}{2} = \frac{12}{24}$ and $\frac{7}{12} = \frac{14}{24}$ so $\frac{13}{24}$ is closer to $\frac{1}{2}$ than is $\frac{7}{12}$. To carry this one step further, we can squelch anyone who suggests $\frac{13}{24}$ as being the next number after $\frac{1}{2}$ by pointing out that $\frac{1}{2} = \frac{24}{48}$ and $\frac{13}{24} = \frac{26}{48}$ so that $\frac{25}{48}$ is more nearly "next to" $\frac{1}{2}$ than is $\frac{13}{24}$. It is clear that this process could be carried on indefinitely and, furthermore, would apply no matter what rational number was involved. That is, we can never identify a "next" rational number after any given rational number. A similar argument would show that we cannot identify a number "just before" a given rational number.

A number line with a very large unit is shown to illustrate the process we went through in searching for the number "next to" $\frac{1}{2}$.

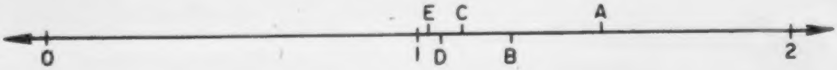


Another way of expressing what we have been talking about is to say that between any two rational numbers, there is always a third rational number; in fact, there are more rational numbers than we could count. Mathematicians sometimes describe this by saying that the set of rational numbers is dense. The word is not important to us, but is descriptive of the packing of points representing rational numbers closer and closer together on the number line. Although we can visualize that the points representing the rational numbers are densely packed, there are many points on the number line whose coordinates are not rational numbers.

Many points are associated with numbers such as π , $\sqrt{2}$, $\sqrt[3]{7}$, and so on. We are not going to consider such numbers in this text, but we mention them to indicate that the number line is not yet complete. There is a point associated with every rational number but there is not a rational number for every point.

PROBLEMS

14. Name the rational numbers associated with the points A , B , C , D, and E below, where A is halfway between 1 and 2 , B halfway between 1 and A , etc.



15. How many numbers are there between 1 and the number associated with point E ?

SUMMARY

Every nonnegative rational number can be represented by many different fractions of the form $\frac{a}{b}$, where a designates a whole number and b designates a counting number. All fractions for the same rational number are said to be equivalent. The problems of changing a fraction to "higher terms" or to "lower terms" or to lowest terms are essentially problems of renaming. In this connection we use to advantage the fact that

$$\frac{a}{b} = \frac{a \times k}{b \times k} \quad \text{(where } k \text{ designates a counting number)}$$

and also the fact that

$$\frac{a}{b} = \frac{a \div k}{b \div k} \quad \text{(where } k \text{ designates a factor of } a \text{ and } b \text{).}$$

Equality and order among the nonnegative rational numbers can be established on the basis of these conditions:

$$\frac{a}{b} = \frac{c}{d} \quad \text{if and only if} \quad a \times d = b \times c .$$

$$\frac{a}{b} > \frac{c}{d} \quad \text{if and only if} \quad a \times d > b \times c .$$

$$\frac{a}{b} < \frac{c}{d} \quad \text{if and only if} \quad a \times d < b \times c .$$

We have seen that a rational number may be used to designate the quotient of any whole number, a , and any counting number, b :

$$a \div b = \frac{a}{b} .$$

Finally, we have pointed to the fact that between any two rational numbers, no matter how close they are to each other, there are many other rational numbers. Among other things this means that, unlike the whole numbers, one cannot identify the number that comes "just before" or "just after" a given rational number.

APPLICATIONS TO TEACHING

We have emphasized the use of several different models in developing ideas about rational numbers:

a. unit regions (plane and solid), partitioned into congruent regions;

b. unit segments, partitioned into congruent segments; and

c. unit arrays (or sets), partitioned into equivalent subsets.

Children encounter each of these models in connection with their everyday experiences, such as:

a. displaying a fractional part of a candy bar,
b. displaying a fractional part of a piece of string,
c. displaying a fractional part of a bag of marbles.

It is important that children have ample experience with each of the models identified if children are to be able to apply rational numbers correctly and effectively. Variety of representation is imperative in this connection.

QUESTION

"Why do you not insist on changing improper fractions to mixed numerals?"

The plea is not so much to have numbers expressed in one form or another as it is to have the pupils realize that the various forms are names for the same numbers. Sometimes it is more convenient to have the mixed numeral form than the improper fraction form. Sometimes it is the other way around.

For example, an answer of $\frac{15}{2}$ for the number of pounds of candy is more meaningfully stated as $7\frac{1}{2}$ pounds. If more computations need to be made on this answer, say, to find the price at 37¢, it would be pointless to express $\frac{15}{2}$ as $7\frac{1}{2}$ and change it again to $\frac{15}{2}$ to obtain the product.

VOCABULARY

Array *
Common Denominator
Congruent Segments
Denominator *
Dense
Division *
Equivalent Fractions *
Fraction *
Fractional Numbers *
Greater than *
Higher Terms
Least Common Denominator
Lower Terms
Mixed Form *
Mixed Numeral *
Nonnegative Rational Number
Numerator *
Rational Number *
Rectangular Region
Simplest Form *
Solid Region *
Square Region
Unit *
Unit Region *
Unit Set

EXERCISES - CHAPTER 13

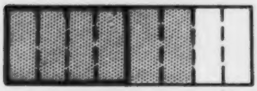
1. Using rectangular regions as your unit regions, represent each of the following by partitioning the units and shading in parts.

a.	$\frac{3}{4}$	e.	$\frac{7}{5}$
b.	$\frac{2}{4}$	f.	$\frac{0}{3}$
c.	$\frac{4}{4}$	g.	$\frac{9}{4}$
d.	$\frac{5}{4}$	h.	$\frac{1}{7}$

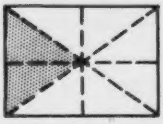
2. Using unit segments on number lines, represent each of the fractions a - h of Exercise 1.

3. Using arrays or equivalent sets, represent each of the fractions a - h of Exercise 1.

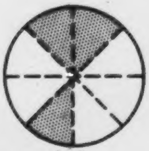
4. Most of the following figures are models for rational numbers. Some of them are not models because the unit has not been partitioned into congruent parts. For each one that is a proper model, give the rational number which is pictured.



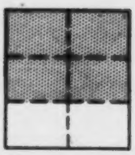
(a)



(b)

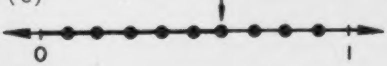


(c)

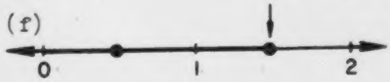


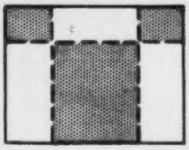
(d)

(e)



(f)





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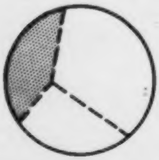

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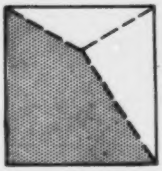
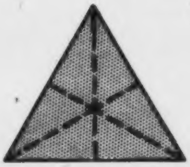


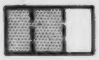
(i)

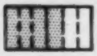


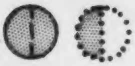
(j)

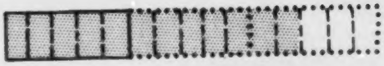
SOLUTIONS FOR PROBLEMS

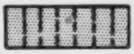
1. Many models may be used. These are illustrative only.

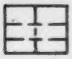
a.  $\frac{2}{3}$

b.  $\frac{4}{6}$

c.  $\frac{3}{2}$

d.  $\frac{12}{5}$

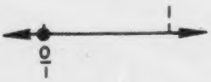
e.  $\frac{7}{7}$

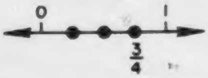
f.  $\frac{0}{6}$

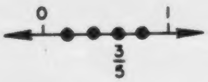
2. The figures are not good models because they are not partitioned into congruent regions.

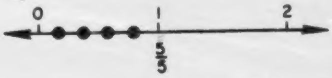
3. a. $\frac{2}{4}$ b. $\frac{2}{2}$ c. $\frac{7}{4}$ d. $\frac{0}{6}$

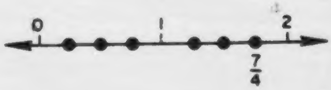
4.

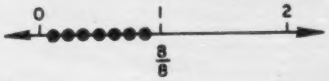
a. 

b. 

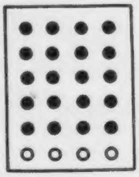
c. 

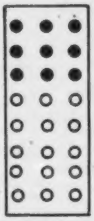
d. 

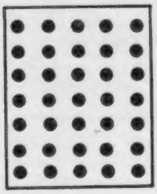
e. 

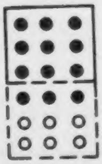
f. 

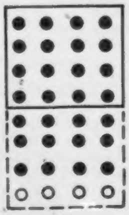
5. These models are illustrative only.

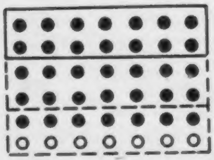
a. 

b. 

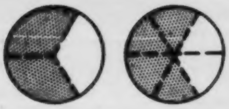
c. 

d. 

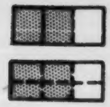
e. 

f. 

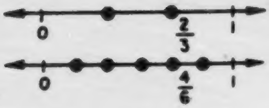
6. For example:



or



or


0
$\frac{2}{3}$
1
0
$\frac{4}{6}$
1

7. a. 8 b. 28 c. 7

8. a. $k = 4$ b. $k = 3$ c. $k = 21$

9.

	Higher terms; many answers, e.g.:	Lower terms; any of these:	Lowest terms:
a.	$\frac{48}{72}$, $\frac{72}{108}$, $\frac{240}{360}$, etc.	$\frac{12}{18}$, $\frac{8}{12}$, $\frac{6}{9}$, $\frac{4}{6}$, $\frac{2}{3}$	$\frac{2}{3}$
b.	$\frac{60}{120}$, $\frac{180}{240}$, $\frac{240}{480}$, etc.	$\frac{15}{30}$, $\frac{10}{20}$, $\frac{6}{12}$, $\frac{5}{10}$, $\frac{3}{6}$, $\frac{1}{2}$	$\frac{1}{2}$

10. Since in $\frac{a}{b} = \frac{a \times k}{b \times k}$ $\underline{k}$ can be any counting number, there is no limit to how large the numerator and denominator can become.

11. a. $f = 15$, $\frac{30 \div 15}{45 \div 15} = \frac{2}{3}$ c. $f = 13$, $\frac{39 \div 13}{52 \div 13} = \frac{3}{4}$

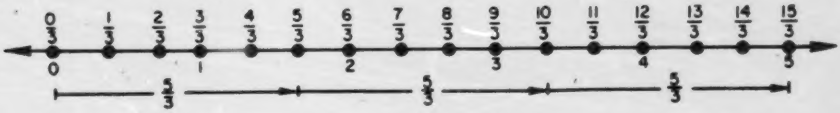
b. $f = 12$, $\frac{24 \div 12}{36 \div 12} = \frac{2}{3}$

12. a. $\frac{6}{14} < \frac{7}{16}$ b. $\frac{6}{8} = \frac{9}{12}$ c. $\frac{30}{63} < \frac{15}{28}$

d. $\frac{3}{4} > \frac{36}{52}$ e. $\frac{9}{20} = \frac{45}{100}$ f. $\frac{143}{13} > \frac{1043}{103}$

13. a. $n = 5 \div 3 = \frac{5}{3}$

b.

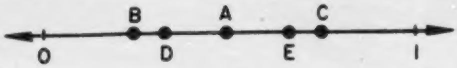

$\frac{0}{3}$
$\frac{1}{3}$
$\frac{2}{3}$
$\frac{3}{3}$
$\frac{4}{3}$
$\frac{5}{3}$
$\frac{6}{3}$
$\frac{7}{3}$
$\frac{8}{3}$
$\frac{9}{3}$
$\frac{10}{3}$
$\frac{11}{3}$
$\frac{12}{3}$
$\frac{13}{3}$
$\frac{14}{3}$
$\frac{15}{3}$
0
1
2
3
4
5
$\frac{5}{3}$
$\frac{5}{3}$
$\frac{5}{3}$

14.

A	B	C	D	E
$\frac{3}{2}$ (or $1\frac{1}{2}$)	$\frac{5}{4}$ (or $1\frac{1}{4}$)	$\frac{9}{8}$ (or $1\frac{1}{8}$)	$\frac{17}{16}$ (or $1\frac{1}{16}$)	$\frac{33}{32}$ (or $1\frac{1}{32}$)

15. More than can be counted (actually "infinitely many").

5. Consider the points labeled A , B , C , D and E on the number line:



a. Give a fraction name to each of the points.

b. Is the rational number located at point B less than or greater than the one located at D ? Explain your answer.

c. In terms of the marks on this number line, what two fraction names could be assigned to the point A ?

6. Interpret on the number line the following:

a. $\frac{20}{5} = 4$ b. $\frac{20}{4} = 5$ c. $\frac{23}{5} = 4\frac{3}{5}$

7. Show on the number line the equality:

$$\frac{2}{8} = \frac{3}{12}$$

8. Tell which of the following fractions are in "simplest form".

$\frac{6}{12}$, $\frac{11}{4}$, $\frac{7}{12}$, $\frac{12}{13}$, $\frac{510}{513}$, $\frac{7}{412}$, $\frac{412}{7}$, $\frac{10}{12}$, $\frac{13}{26}$, $\frac{2}{3}$

9. For each pair of rational numbers named below, indicate whether the first is equal to the second, greater than the second, or less than the second.

a. $\frac{1}{25}$, $\frac{1}{24}$ c. $\frac{7}{8}$, $\frac{5}{6}$ e. $\frac{13}{26}$, $\frac{9}{18}$

b. $\frac{11}{24}$, $\frac{12}{26}$ d. $\frac{17}{32}$, $\frac{1}{2}$

10. Express each of these in mixed form.

a. $\frac{7}{4}$ b. $\frac{15}{8}$ c. $\frac{21}{9}$ d. $\frac{34}{15}$ e. $\frac{56}{12}$

Chapter 14

PREMEASUREMENT CONCEPTS

INTRODUCTION

Certain basic geometric figures and concepts have been presented in Chapters 5 and 11. Recall that common physical objects provided the foundations on which the development was built. It was done this way because this is the way in which geometrical ideas are conveyed to young children. Little was said at that time about geometric solids. This topic will now be extended to gain familiarity with associated vocabulary and characteristics as has already been accomplished for many plane figures.

The notion of congruence which has appeared in the earlier discussion will also be a vital concept in the following development. It will provide a means of ordering sets of points which will in turn lead to the concept of measure. By this, we do not mean ordering the points as we have done on the number line. We mean assigning an order to sets of points as for example, among various segments, among plane regions, or solid regions. The corresponding measures are for lengths, areas, and volumes. Thus, we can compare the "sizes" of different geometric objects. The concept of measure will be discussed in Chapter 16. In this chapter, we want first to identify some of the geometrical relationships and configurations by their mathematical names and next, to clarify the concept of ordering sets of points.

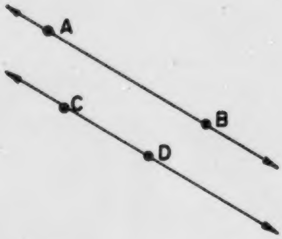
INTERSECTING AND PARALLEL

The terms intersecting and parallel are familiar though common usage in describing physical phenomena. We speak of a road that runs parallel to a railroad track, or we speak of the intersection of Polk and Fell Streets, and so on. These everyday references describe, although somewhat more loosely, the same relationships that the terms imply in geometry.

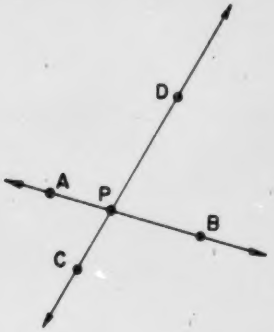
Recall that intersection is one of the set operations dealt with earlier. The intersection of two sets yields a set whose members are those which the two sets have in common. The intersection of two sets, then, can be the emtpy set or it can have members; it is the empty set if the two are disjoint.

Thinking again of an example of streets, if First Street and Second Street run parallel, there is no intersection. Technically, we would simply say the intersection is empty. However, the less formal description, that "there is no intersection", is often used in geometry for the more accurate description, "the intersection is empty".

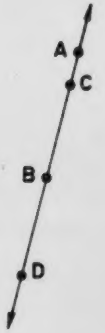
Consider the lines $\overleftrightarrow{AB}$ and $\overleftrightarrow{CD}$ as our two sets of points. The operation of intersection may yield the empty set, a single point, or a line. The drawings illustrate these possible situations.



$\overleftrightarrow{AB} \cap \overleftrightarrow{CD} = \{\ \}$



$\overleftrightarrow{AB} \cap \overleftrightarrow{CD} = P$

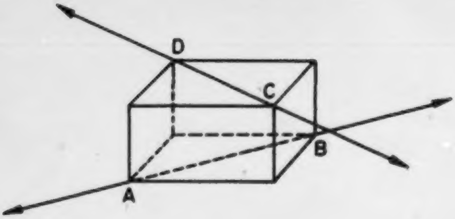


$\overleftrightarrow{AB} \cap \overleftrightarrow{CD} = \overleftrightarrow{AB}$

In general, "do intersect" or simply "intersects" implies the intersection has members; "do not intersect" implies the intersection is empty.

Although we have only used lines as examples, any sets of points can be considered from the point of view of whether they do or do not intersect. A line may intersect a plane in a line, a point, or not at all; if there is no intersection, the line is said to be parallel to the plane. Two planes may intersect in a line, a plane, or not at all; if they do not intersect, they are said to be parallel.

In space, it is possible that two lines are not parallel and still do not intersect. Picture a road which passes under a railroad bridge. The bridge is not parallel to the road, but does not intersect the road. $\overleftrightarrow{CD}$ and $\overleftrightarrow{AB}$ in this drawing provide another example of nonintersecting, nonparallel lines, $\overleftrightarrow{CD}$ is not parallel to $\overleftrightarrow{AB}$; neither do the two lines intersect.



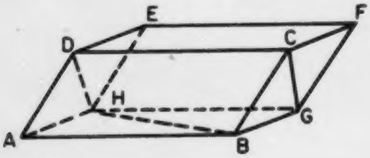
Parallelism for lines may be stated:

> TWO LINES ARE PARALLEL IF THEY LIE IN THE SAME PLANE AND DO NOT INTERSECT.

If S and T are sets of points, certain subsets of S and T may be said to be parallel when S and T are parallel. For example, two segments are parallel if they are subsets of parallel lines. Also, two regions are parallel if they are subsets of two parallel planes. A line may be parallel to a plane, and so on. Note that $\overleftrightarrow{CD}$ and $\overleftrightarrow{AB}$ in the above drawing are subsets of parallel planes but are not considered to be parallel. Lines not lying in the same plane are said to be skew; their intersection is empty. Note also that a plane and a point that is not in the plane may be subsets of parallel planes, but we do not say that the point is parallel to the plane.

PROBLEMS*

1. Identify the intersections of the geometrical figures named. They refer to the drawing. If the intersection is the empty set, state whether the figures are parallel or not.



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*Solutions to problems in this chapter are on page 265.

a. $\overline{CD}$ and $\overline{FC}$

b. the plane region with vertices C, D, E, F and the plane region with vertices A, D, E, H.

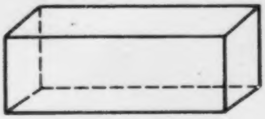
c. $\overline{DH}$ and $\overline{CG}$

d. $\overline{EH}$ and the plane region with vertices A, B, G, H

e. $\overline{BH}$ and $\overline{EF}$

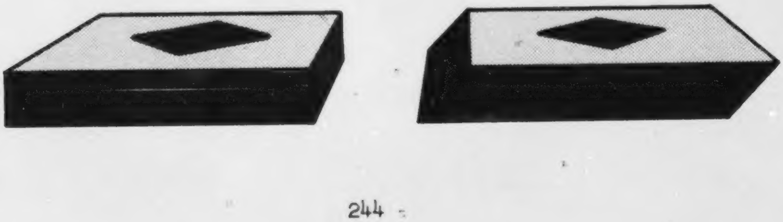
PRISMS

In Chapter 5, a rectangular prism was identified and looked at briefly. It was noted that it was composed of six plane regions called faces. The intersection of any two faces may be empty. If two faces "do intersect", however, their intersection is a segment called an edge. In the same manner, intersecting edges determine a point called a vertex. Thus, the above rectangular prism is the union of its six faces, contains twelve edges and eight vertices. Its shape was abstracted from a rectangular box; all of its faces are rectangular regions.

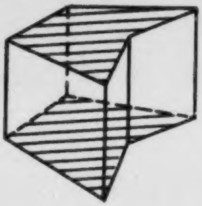


The pictures below of a deck of cards pushed into an oblique position is also a model of a rectangular prism. The criteria for a prism are simply

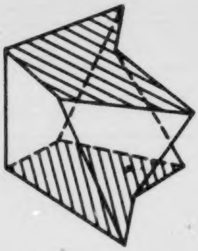
> there are two congruent polygonal regions lying in parallel planes, and the edges which do not belong to these parallel planes are all parallel to one another.



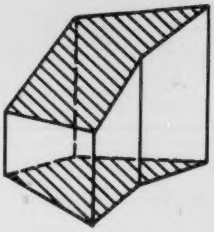
Thus in the figures below, the first is a prism but the other two are not.



Congruent polygonal regions in parallel planes; edges parallel.

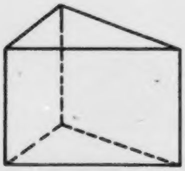


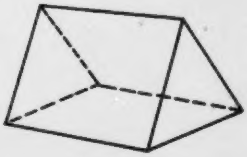
Congruent polygonal regions in parallel planes; edges not parallel.*



Edges parallel; polygonal regions not congruent.ǂ

The congruent regions in the parallel planes are called bases of the prism, and the prism may be identified according to the kind of bases it has. For example, the rectangular prism has rectangular regions for bases; the prism shown in the figure at the left above is a pentagonal prism; either of the figures below is a triangular prism.





The faces of a prism that are not bases are called the lateral faces. Note that each lateral face is a parallelogram region; the boundary of each lateral face consists of two parallel edges called lateral edges and two sides of congruent polygons. The two sides of the congruent polygons are also parallel, thus the boundary of each lateral face is a parallelogram.

If the bases of a prism are also parallelogram regions, the prism is called a parallelepiped. Thus, the rectangular prisms are a subfamily of the parallelepipeds. A cube, which is the union of six congruent square regions, is another kind of specialized rectangular prism and, hence, is also a parallelepiped. A generic chain of quadrilateral prisms can thus be formed just as was identified for quadrilaterals.

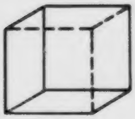
---

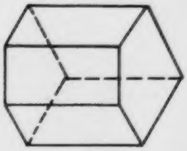
*Imagine the top of the first figure given a twist.
ǂImagine cutting the first solid at a slant to the base.

The above two pictures of the deck of cards illustrate another property by which prisms are classified. In the first case, the lateral faces are rectangular regions; in the second drawing they are parallelogram regions only. The first is a right prism; the second is an oblique prism. The lateral faces of right prisms are rectangles. The triangular prisms shown above are right prisms. A cube is a right prism all of whose faces are rectangular regions and, more specifically, are square regions.

PROBLEMS

2. a. Select the figures which represent prisms and give the name which best describes each.
   b. For those figures which do not represent prisms, state why they fail to qualify.


(A)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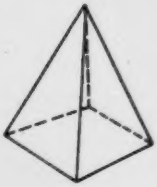
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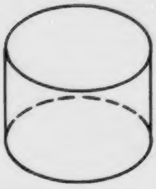
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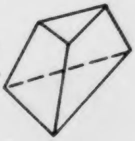
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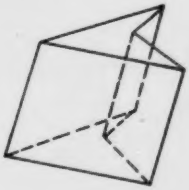
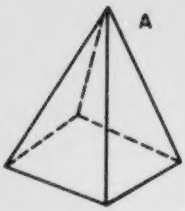
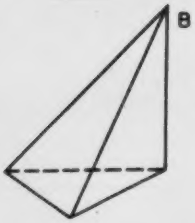
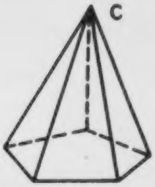

(F)

3. Draw a figure representing an oblique square prism.

PYRAMIDS







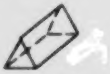
The drawings above represent examples of a familiar set of geometric solids, namely pyramids. As is the case for prisms, there are a great variety of sizes and shapes of pyramids. Each must satisfy these criteria:

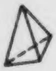
1. there is a polygonal region called the base;
2. there is a point called the apex not in the same plane as the base where all the lateral edges intersect;
3. each lateral face is a triangular region determined by the apex and a side of the base.

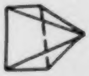
Analogous to the classification of prisms, a pyramid is identified by its base. In the first figure above the base is a square region, and so it is a square pyramid. The others are a triangular pyramid and a pentagonal pyramid respectively. A, B, and C denote their respective apexes.

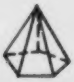
PROBLEMS

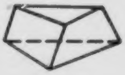
4. Which of the following are drawings of pyramids?

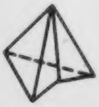




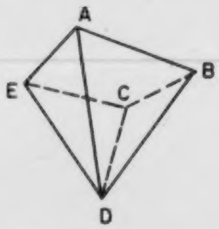








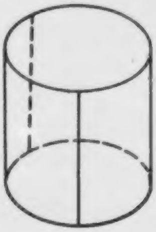
5. a. State an appropriate name for this pyramid.
   b. Identify the apex.
   c. How many edges does it have?
   d. How many faces does it have?

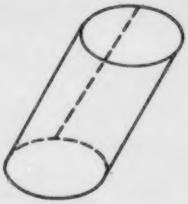


6. What are the possible intersections of two lateral faces of a pyramid?

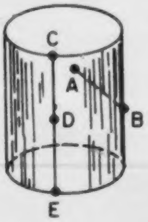
CYLINDERS AND CONES

Although we have not discussed all geometric solids that are the union of flat surfaces, we shall now turn our attention to solids with non-flat surfaces. These two figures represent cylinders. The two faces must be congruent regions in parallel planes. They are called bases of the cylinder, which is consistent with the other uses of the same term. Although the examples show cylinders with circular bases, this is not a requirement of cylinders in general. At this time we shall not consider cylinders with bases of other configurations, so the discussion will be limited to circular cylinders. The boundaries of the congruent bases are then congruent circles and are edges of the cylinder.



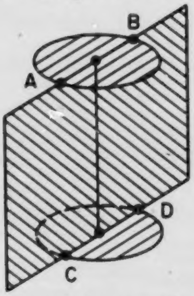


The remaining rounded portion of the simple closed surface which defines the cylinder is its lateral surface. The distinguishing characteristic of a surface which is not flat is that a segment determined by two of its points is not necessarily a subset of the surface. The drawing below illustrates this feature; $\overline{AB}$ is not a subset of the lateral surface of the cylinder. In fact all points of $\overline{AB}$ between A and B are in the interior of the cylinder.



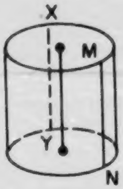
It is possible to find segments which are subsets of the lateral surface of a cylinder, however, such as $\overline{CD}$. In fact, this is a means by which the lateral surface is specified, as we shall show below.

Each of the bases has a center; therefore a segment is determined by these two points. The line containing this segment may be referred to as the line of centers. Consider any plane of which this segment is a subset. It will intersect the bases in two segments called diameters, such as $\overline{AB}$ and $\overline{CD}$ in the figure. Each endpoint of one diameter is to be paired with the appropriate endpoint of the other diameter in order to be able to describe the set of points in the lateral surface. The "appropriate" endpoints of the respective diameters are those which determine a segment that does not intersect the line of centers. Thus, in the drawing, A is paired with C and B is paired with D .

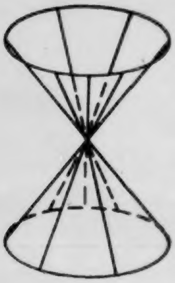


By considering a different plane, we will obtain two new pairs of points. If all such planes are conceived, all such pairs are generated. Then we say we have defined a correspondence between the points in the boundaries of the two congruent bases. Any two points which are thus paired are corresponding points.

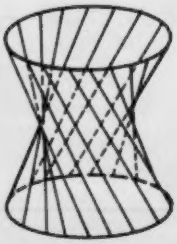
Each of these pairs of corresponding points determines a segment parallel to the segment connecting the centers. The union of all segments determined by corresponding points is the set of points in the desired surface. Each segment is said to be an element of the cylinder. Any two elements are parallel. In the figure, $\overline{MN}$ and $\overline{XY}$ are elements and therefore are parallel.



The preceding description for generating the lateral surface is rather involved. This is because we want to specify the particular correspondence we have in mind since other possible configurations can be formed with the required bases. If a different correspondence were defined between the points of the boundaries, a figure as in (a) and (b) below might evolve.



(a)



(b)

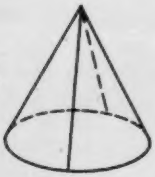
We can now state that a circular cylinder must satisfy these criteria:

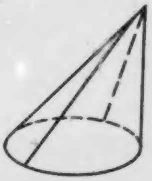
1. there are two congruent circular regions in parallel planes;
2. there is a surface which is the union of all segments determined by corresponding points of the boundaries of the bases.

Referring back to our first two examples of cylinders in this section, the first is a right circular cylinder; the second is oblique. In order to be a right circular cylinder, every element of the cylinder must form right angles with each segment of a base which intersects it.

It is apparent on reflection that there is a distinct similarity between the cylinder and the prism. They each have congruent regions in parallel planes for bases. If an appropriate correspondence were set up between the points of the sides of the bases of a prism, and if line segments joining them were considered such that they are parallel, then the lateral faces would be specified. In fact, the only difference is that the bases of a prism must be polygonal regions while those of a circular cylinder must be circular regions. It is the case that a cylinder can be defined in such a way as to include prisms as a subfamily of cylinders; however, this will not be done for the elementary level.

By the same token that cylinders are analogous to prisms, cones are analogous to pyramids. As with cylinders, we will restrict the plane region of a cone to a circular shape and designate it as the base of the cone. The point which is not in the same plane as the base describes the apex. The lateral surface is not so difficult to describe in this figure. It is simply the set of line segments determined by the apex and each point of the circular boundary of the base.





PROBLEMS

7. State a definition of cylinders so that prisms would be a subfamily of cylinders, namely polygonal cylinders.

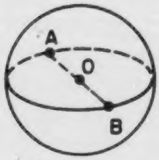
8. Describe or draw representations of the intersections of a plane and a right circular cylinder if the plane does intersect the cylinder and is
   a. parallel to the bases;
   b. parallel to the line of centers;
   c. not parallel to the base nor the line of centers.

SPHERES

The final solid to be included is the sphere. As is the case for a circle, a sphere has a center. All segments connecting the center of the sphere and a point on the sphere are congruent. Indeed, this specifies the set of points in the sphere. They are:

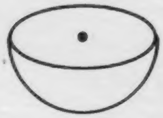
all endpoints of congruent segments
which have one endpoint in common,
but not including their common endpoint.

The congruent segments are radii (singular: radius). The union of two radii which are each subsets of the same line is a diameter. In the figure, O is the center, $\overline{AO}$ and $\overline{OB}$ are radii and therefore congruent,



and $\overline{AB}$ is a diameter.

A hemisphere is half of a sphere. Any plane that contains the center of a sphere will "cut off" a hemisphere.



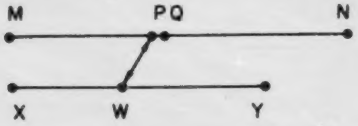
PROBLEMS

9. Identify the intersection of
    a. a plane and a sphere;
    b. the center and the sphere;
    c. a diameter and the sphere;
    d. the center of the sphere and one of its hemispheres.

ORDERING SETS OF POINTS

The ordering of sets is not a new topic. Chapter 2 was devoted to the comparison of sets according to order and certain properties of ordered sets. The approach taken was to pair the members of the two sets in question. Then it was possible to decide whether one set had more or fewer members than the other or whether the two sets were equivalent. If we try to use the same process with sets of points, difficulties are encountered which make the procedure impossible.

Take, for example, two segments, $\overline{MN}$ and $\overline{XY}$. Each is an infinite set, and therefore if we began pairing points we would never exhaust the



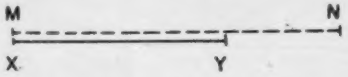
points of either set. This alone eliminates pairing as a means of ordering.

Then, how are segments, and sets of points in general, ordered? We can resort to our concept of congruence to assist us. It has been established intuitively that two line segments are congruent if a movable copy of one can be matched and fitted exactly on the other. A similar procedure serves to indicate whether curves, polygons, plane regions and so on are congruent. It does not prove useful in determining whether or not solid figures are congruent, however, since a movable copy of a solid cannot always be matched and fitted exactly on the other intact. For example, a solid block cannot be fitted into another solid block.

If two sets of points are not congruent, we can still conceive of an order between them. Suppose you measure the dimensions of this book. Its length is shorter than one yard. You are essentially carrying out a comparison of set size with the aid of a movable copy. The sets being compared are an edge of your book and a standard yard defined by the United States Bureau of Standards in Washington, D. C. The movable copy is a yard stick and its scale is a record of the length of the standard. By stating that the length of the edge of the book is shorter than one yard, we are ordering the sizes of two physical representations of line segments. In particular, you book is shorter than the standard yard.

Geometrical segments are handled in a similar fashion. Suppose it is desired to order the two sets $\overline{MN}$ and $\overline{XY}$. We make a copy of $\overline{MN}$, indicated by the dotted segment, and lay it over $\overline{XY}$. We have already

said that if they fit exactly, then $\overline{MN}$ and $\overline{XY}$ would be congruent. If, however, they do not, one of two situations must exist. $\overline{XY}$ will be congruent to a proper subset of $\overline{MN}$ or $\overline{MN}$ will be congruent to a proper subset of $\overline{XY}$. In the first instance, we would say $\overline{XY}$ *is shorter than* $\overline{MN}$ or, equivalently, $\overline{MN}$ *is longer than* $\overline{XY}$. The second possibility is interpreted as $\overline{MN}$ is shorter than $\overline{XY}$ or $\overline{XY}$ is longer than $\overline{MN}$. Our example demonstrates the first case, since $\overline{XY}$ is congruent to a proper subset of $\overline{MN}$. We can order the sets by



$\overline{XY}$, $\overline{MN}$ in increasing order.

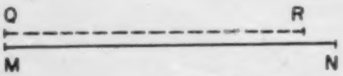
For finite sets, A and B , recall that comparing sets assured exactly one of three possible outcomes:

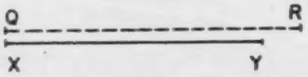
A is equivalent to B ;
A has more members that B ;
A has fewer members than B .

Now we can state the parallel relationships for infinite sets of points, $\overline{AB}$ and $\overline{CD}$:

$\overline{AB}$ is congruent to $\overline{CD}$;
$\overline{AB}$ is longer than $\overline{CD}$;
$\overline{AB}$ is shorter than $\overline{CD}$.

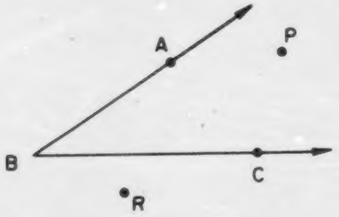
Note that "$\overline{AB}$ is longer than $\overline{CD}$" does *not* mean $\overline{AB}$ has more members than $\overline{CD}$. We are saying nothing about "how many" in relating infinite sets. By repeated comparison, it is possible to order more than two segments. Thus $\overline{QR}$ below would fit into the order $\overline{XY}$, $\overline{QR}$, $\overline{MN}$ as the diagram illustrates. We find that $\overline{QR}$ is congruent to a subset of $\overline{MN}$, and that $\overline{XY}$ is congruent to a subset of $\overline{QR}$. Therefore $\overline{QR}$ is shorter than $\overline{MN}$ and $\overline{QR}$ is longer than $\overline{XY}$.

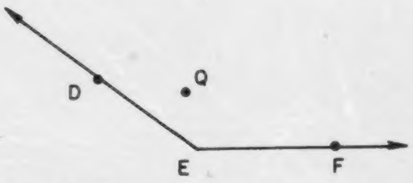




In Chapter 16, these order relationships will be restated in terms of numbers associated with segments. These numbers will be the measures of the segments. By our ordering, however, we have done no measuring.

The second kind of geometric figures that we wish to order is angles. An angle is the set of points defined by the union of two rays which have a common endpoint and which are not subsets of the same line. Just as simple closed curves separate a plane into three subsets (the curve, its interior and its exterior), angles can be thought of as doing the same thing. A point is in the interior of an angle if it lies between two



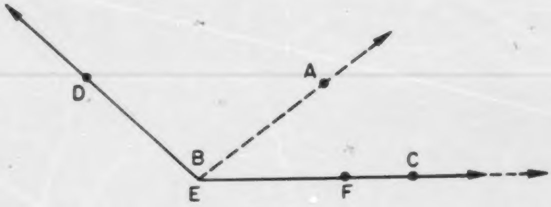


points, one on each ray, exclusive of the vertex. Thus P is in the interior of $\angle ABC$ and Q is in the interior of $\angle DEF$. P is in the exterior of $\angle DEF$ and R is an exterior point of $\angle ABC$.

To order two angles, we rely on a movable copy of one in much the same manner as we did for segments. For the angles pictured above, we could place a copy of $\angle ABC$ over $\angle DEF$ so that one side of the copy coincides with one side of $\angle DEF$. The figure below shows one way the copy can be positioned. If the second side of the copy also coincides with the second side of $\angle DEF$, we would say

$\angle ABC$ is congruent to $\angle DEF$.

(See Chapter 11 for a discussion of congruent angles).



If it is not possible to get such a coincidence, as it is not for the ones pictured above, we define an order. Note that the points of $\overleftrightarrow{BA}$, except the endpoint B, lies in the interior of ∠DEF. Whenever this phenomenon holds, we say

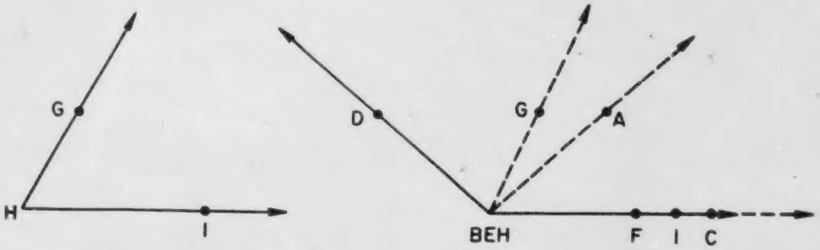
∠ABC is smaller than ∠DEF

or, equivalently, ∠DEF is larger than ∠ABC . If it happened that the interiors of the two angles have points in common and that $\overrightarrow{BA}$, except for B , were a subset of the exterior of ∠DEF , then

∠ABC is larger than ∠DEF

or ∠DEF is smaller than ∠ABC .

Considering a third angle, ∠GHI , we find that $\overrightarrow{GH}$, except for G , lies in the exterior of ∠ABC and in the interior of ∠DEF . Two statements expressing this are ∠GHI is larger than ∠ABC , and ∠GHI is smaller than ∠DEF . In increasing order, we could write ∠ABC , ∠GHI , ∠DEF . As for segments, this procedure can be repeated indefinitely for as many angles as we wish. Congruent angles would occupy the same position in the order.

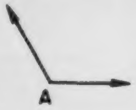


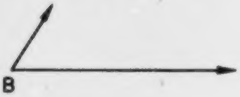
The definition of measurement for angles will not be included in Chapter 16 because it is not treated in the K-1 text materials. It has been discussed here to indicate that the ordering of sets of points can be accomplished for figures other than segments. It is actually possible to use congruence as a means of ordering regions and solids also, although it is a bit more complicated. It is not possible, however, to order unlike sets of points; that is, we cannot order segments and angles, nor segments and plane regions, and so on.

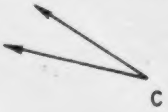
PROBLEMS

10. Represent $\overline{AB}$, $\overline{CD}$ and $\overline{EF}$ such that their order from shortest to longest is $\overline{CD}$, $\overline{AB}$, $\overline{EF}$.

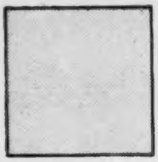
11. Place the sets represented by the angles below in increasing order.

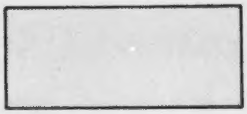






12. Can you devise a means of ordering the two regions shown below?





APPLICATIONS TO TEACHING

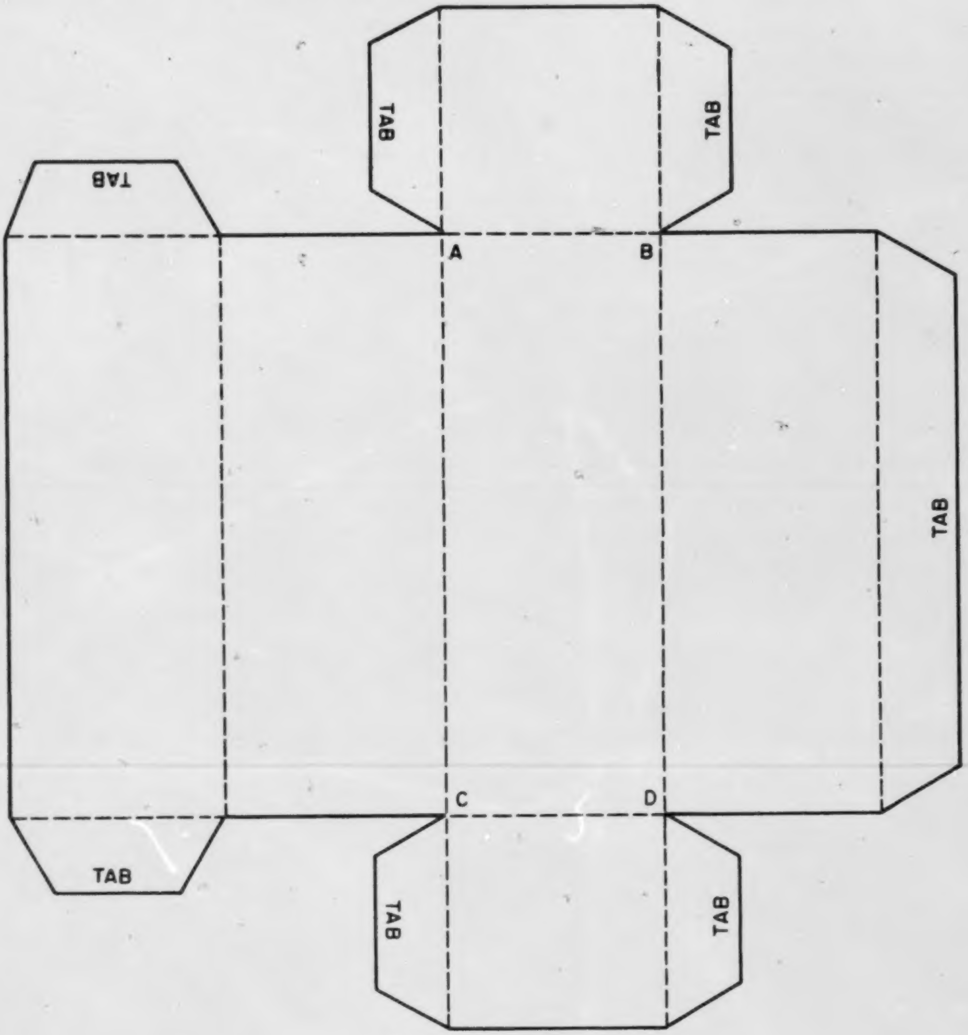
Teachers have found it most helpful to have in the room a wide collection of objects which illustrate geometrical solids. Children also enjoy bringing such objects from home. Effective ways of using these and other models have been recommended in this section of Chapter 5.

On the next pages are included four patterns to be used in constructing geometrical solids out of paper. Having the children observe your demonstration of a construction emphasizes two aspects of solids. Many are the union of plane regions that do not lie in the same plane, and they are hollow.

The ideas in the pre-measurement section are most important. The children should be asked to participate as much as possible in manipulating figures to compare their sizes, both to understand congruence and order. They often experience some difficulty in visualizing congruent regions if they have different orientations, so practice should be provided with this in mind.

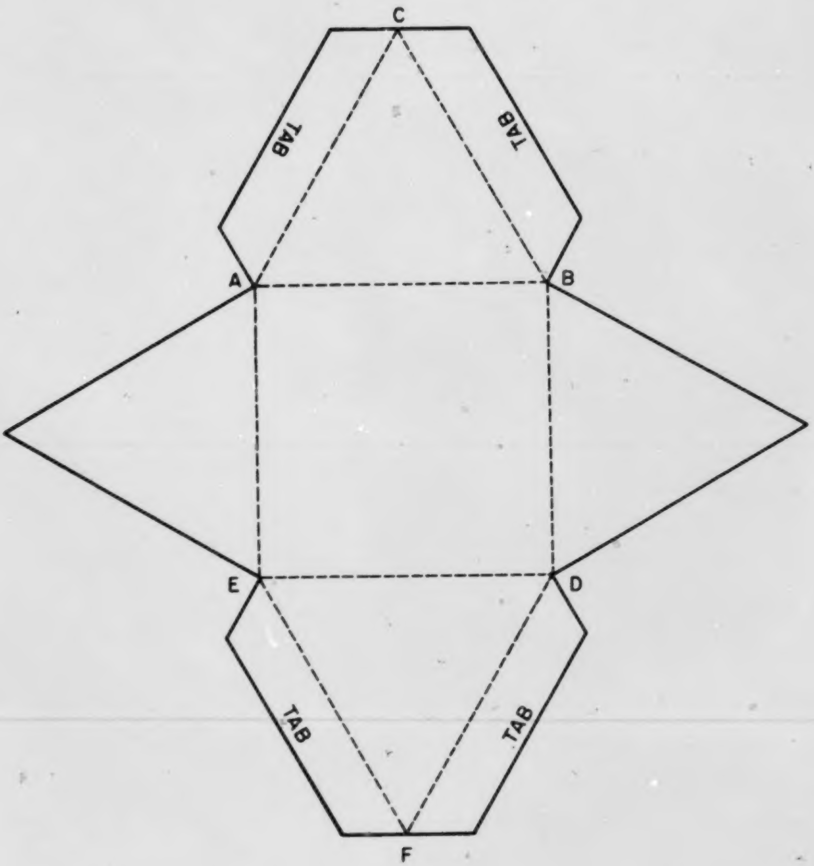
PRISM - Construction of a square prism.

1. Draw a rectangle with vertices A , B , C , D as shown.
2. Draw, as shown, three other rectangles congruent to the rectangle already drawn with tabs.
3. Draw the two squares along $\overline{AB}$ and $\overline{DC}$ with tabs, as shown.
4. Cut around the boundary of the figure and fold along the dashed line segments.
5. Use scotch tape or paste to hold the model together. The tabs will help give rigidity to the model. You may want to trim them some if you use scotch tape.
6. The bases of this rectangular prism are squares, hence the name - square prism.
7. This picture has been reduced photographically. The original had the length of $\overline{AB}$ as 1 1/2" and that of $\overline{BC}$ as 4" . This made a 1 1/2" × 1 1/2" × 4" square prism.



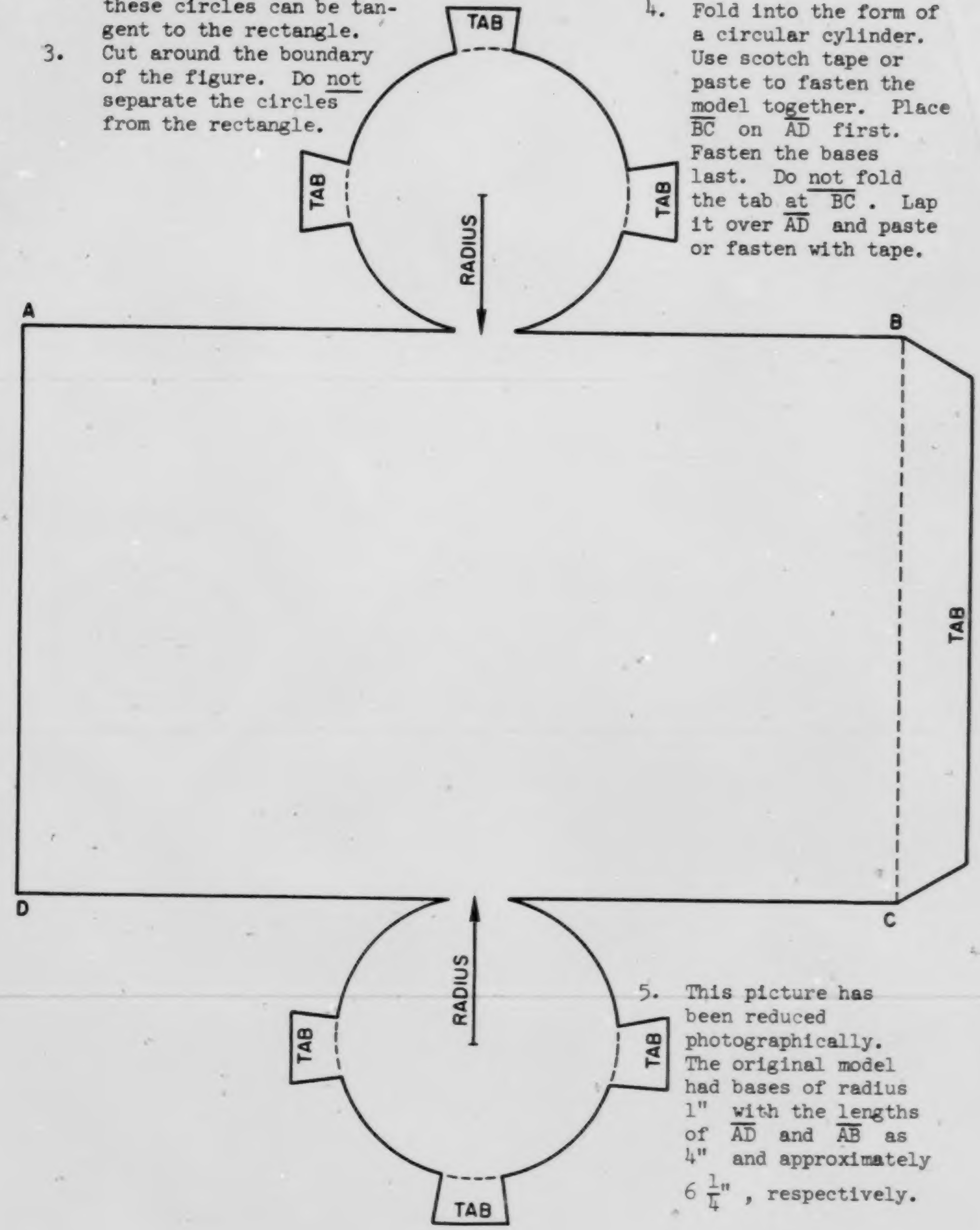
YRAMID - Construction of a square pyramid.

1. Draw a square with vertices A , B , D , E as shown.
2. Draw the arcs with centers at A and B and radius $\overline{AB}$. Label the intersection shown as C .
3. Draw dashed line segments $\overline{AC}$ and $\overline{BC}$ to form "dashed" equilateral triangle with vertices A , B , C . Draw tabs as shown.
4. Repeat step 3 to obtain "dashed" equilateral triangle with vertices E , D , F with tabs as shown.
5. Draw the equilateral triangle shown on $\overline{BD}$ and $\overline{AE}$.
6. Cut around the boundary and fold along the dashed line segments.
7. Fasten with scotch tape or paste. The tabs will help in putting the model together. You may want to trim some of them if you use scotch tape.
8. This picture has been reduced photographically. The original model had the lengths of $\overline{AB}$ as 2" .



CYLINDER - Construction of a circular cylinder.

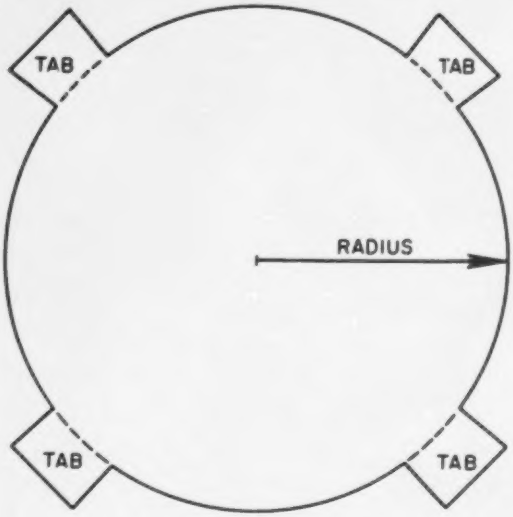
1. Draw the rectangle with vertices A , B , C , D .
2. Draw two congruent circles with radius as shown. In order to make the model easier to construct, these circles can be tangent to the rectangle.
3. Cut around the boundary of the figure. Do *not* separate the circles from the rectangle.
4. Fold into the form of a circular cylinder. Use scotch tape or paste to fasten the model together. Place $\overline{BC}$ on $\overline{AD}$ first. Fasten the bases last. Do *not* fold the tab at $\overline{BC}$. Lap it over $\overline{AD}$ and paste or fasten with tape.

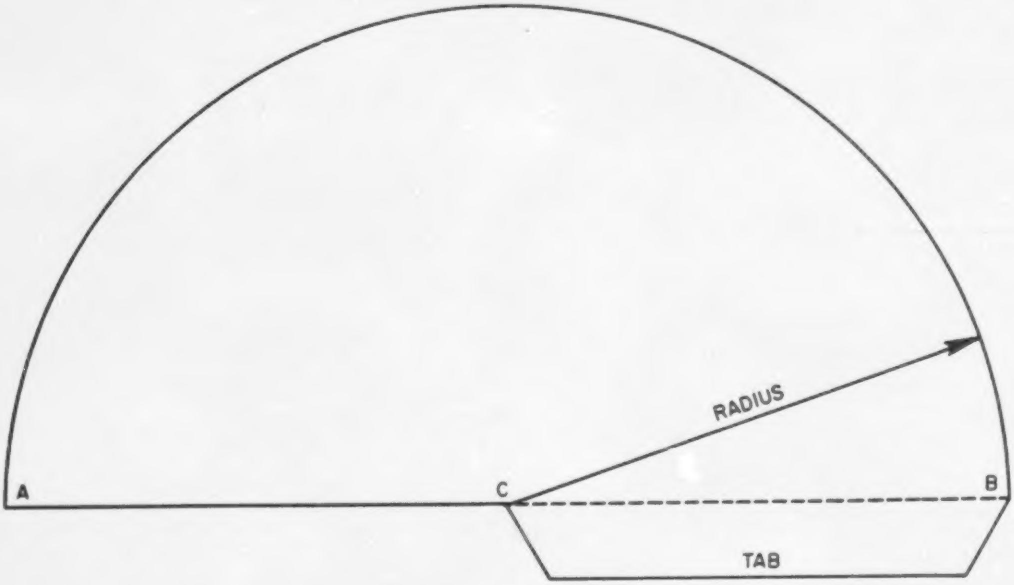


5. This picture has been reduced photographically. The original model had bases of radius 1" with the lengths of $\overline{AD}$ and $\overline{AB}$ as 4" and approximately $6\frac{1}{4}$" , respectively.

CONE - Construction of a circular cone.

1. Use a compass to draw a circle with a radius as shown in the diagram. Draw tabs as shown.
2. Cut around the boundary of this figure. The circular region will be the base of the cone.
3. Use a compass to draw a semicircle with a radius as shown in the diagram. C is the center of the circle. $\overline{AB}$ is a diameter. Draw the tab as shown.
4. Cut around this figure.
5. Fasten $\overline{AC}$ to $\overline{BC}$ with scotch tape or paste so that $\overline{AC}$ falls on $\overline{BC}$.
6. Fasten the base to this model by folding the tabs and using scotch tape or paste.



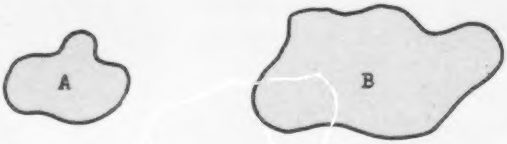


QUESTION

"How do ordering points, ordering sets of points, ordered sets, and ordered pairs differ?"

Ordering points is connected with our development of the number line. Here, with the designation of a particular point as starting point, and with a given segment selected as a unit, the line is marked off at equally spaced intervals and the marked points are associated with the whole numbers in the usual sequence: 0, 1, 2, 3, ... The number line is next filled in by associating points between those named above with other rational numbers such as $\frac{1}{2}$, $\frac{2}{3}$, $\frac{5}{4}$, and so on. In so doing, any two points named in the number line are ordered with the designation of which precedes the other. Consequently, all the points named in the number line are ordered.

Ordering sets of points may be illustrated by taking two sets of points in the plane. Suppose these are represented in A and B below.



We want to convey the notion that agrees with our intuitive sense: that B occupies more space than A. Counting the number of points in each set would not do since each set contains an infinite number of points. A scheme is sought whereby we can still assign a number to each set to indicate an order in "size". Thus, a number evolves as a measure--such as area, volume, length, etc. In so doing, we are ordering sets of points.

The set of rational numbers is an example of an ordered set. The letters in the alphabet are another example of an ordered set. Although, with sets in general, we state that it is immaterial in what order the elements are listed, in an ordered set, for any two members, it is possible to designate which member precedes the other. By association with members of the ordered set of rational numbers, for example, the points in the rational number line become an ordered set. Similarly, sets of points may be ordered via their measure.

An ordered pair arises from the product set. An illustration of an ordered pair may be the element denoted (4, 3) if it is agreed than an order must be observed in the naming of the components 4 and 3. This may be, for example, the designation for the child in the 4th row, 3rd seat. Thus, the notation in the theater ticket stub, D3, may be indicative of an ordered pair. Or, when you say, "Fred and Maggie are president and secretary of the Clamdiggers Society, respectively", an order is induced in the pair, (Fred, Maggie).

VOCABULARY

Apex

Bases (of a geometric figure)*

Center (of a sphere)

Cone

Cube

Cylinder

Edge *

Element (of a cylinder)

Hemisphere

Intersection *

Lateral Edges

Lateral Faces

Lateral Surface

Line of Centers

Parallel

Parallelepiped

Pentagonal Prism

Pentagonal Pyramid

Prism

Pyramid

Radius (of a sphere)

Right Circular Cylinder

Skew Lines

Sphere

Square Pyramid

Triangular Prism

Triangular Pyramid

Vertex (of a prism) *

EXERCISES - CHAPTER 14

1. Why is the following definition of parallel segments not sufficient to determine what we mean by parallel segments?

   > Two segments are parallel if they lie
   > in the same plane and do not intersect.

2. What are the sets which may result in the intersection of a line and a plane?

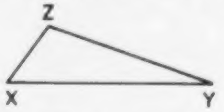
3. Construct a paper model of a square pyramid using the pattern on page

4. a. How many edges does a triangular pyramid have?
   b. How many edges does a rectangular pyramid have?
   c. If the base of a pyramid has n sides, how many edges does the pyramid have?

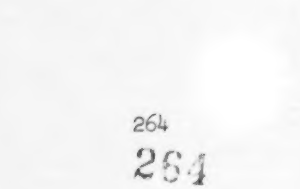
5. Identify by a drawing the intersection of a plane parallel to $\overline{AO}$ and the cone, if A is the apex and O is the center of the base. Assume the plane intersects the cone in more than one point.

6. Which of the following solid regions must be convex sets?
   a. sphere; b. circular cylinder; c. quadrilateral pyramid.

7. State in increasing order the sides of the triangle.



8. Why is it incorrect to say $\overrightarrow{AB}$ is a subset of the interior of $\angle$ MAL ?



SOLUTIONS FOR PROBLEMS

1. a. C b. $\overline{DE}$ c. { } ; they are parallel d. H e. { } ; not parallel.

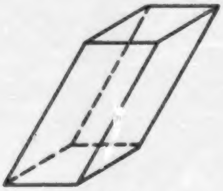
2. a. (A) cube; (B) right pentagonal prism; (F) non-convex quadrilateral prism.

   b. (C) There are not 2 congruent, parallel bases; the lateral edges are not parallel.

      (D) The congruent faces are not polygonal; the lateral surface is not the union of parallelogram regions.

      (E) The parallel bases are not congruent; the lateral edges are not parallel.

3.



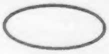
4. (b) , (c) , (d) , (f)

5. a. quadrilateral pyramid
   b. D
   c. 8
   d. 5

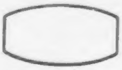
6. A lateral edge or the apex

7. A cylinder is a geometric solid which is the union of two similarly oriented parallel regions whose boundaries are simple closed curves and all the segments determined by corresponding points of the congruent boundaries.

8. a. a circle; b. a rectangle or a segment congruent to the segment connecting the centers; c.



or

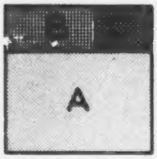


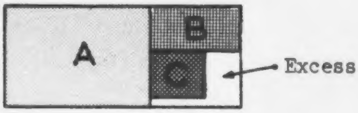
9. a. a circle, a point, or { } ; b. { } ; the center is not part of the sphere; c. two points--the endpoints of the diameter; d. { } .

10. C________D A______________B E____________________F

11. $\angle C$, $\angle B$, $\angle A$

12. We can partition one region, make movable copies and lay them on the other region. If they fit, we will say they have the same size. If they do not, one will be larger than the other.





Thus, the rectangular region is larger than the square region.

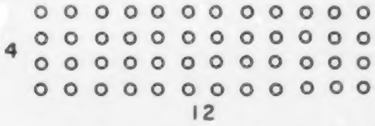
Chapter 15

MULTIPLICATION AND DIVISION TECHNIQUES

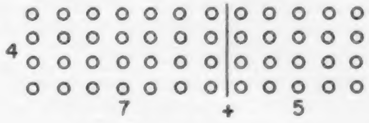
MULTIPLYING NUMBERS GREATER THAN TEN

The ability to compute with understanding and skill when multiplying whole numbers greater than 10 depends upon several things. Among these are: knowledge of basic multiplication facts, ability to use a multiple of 10 as a factor, familiarity with our decimal place value numeration system, and ability to apply multiplication properties (commutative, associative, distributive over addition, etc.).

First let us consider the product of 4 and 12, for which we may display the array



By partitioning the array into two arrays so that each row has less than 10 members, we need to use only basic multiplication facts, the distributive property of multiplication over addition, and addition facts in order to compute the product of 4 and 12. For instance, we may partition the 4 by 12 array into a 4 by 7 array and a 4 by 5 array:



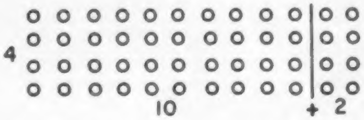
Then, $4 \times 12 = 4 \times (7 + 5)$

$$\begin{aligned} &= (4 \times 7) + (4 \times 5) \\ &= 28 + 20 \\ &= 48 \end{aligned}$$

We have gone directly here from 28 + 20 to 48 and have omitted the intervening steps:

$$
\begin{aligned}
&= 28 + 20 \\
&= 20 + 28 \\
&= 20 + (20 + 8) \\
&= (20 + 20) + 8 \\
&= 40 + 8 \\
&= 48
\end{aligned}
$$

By choosing the numeral $7 + 5$ for 12, only basic multiplication facts from the multiplication table are needed. We could also have chosen to consider 12 as $3 + 9$, $4 + 8$ or $6 + 6$ without the necessity of going outside the table. However, since in terms of our numeration system we commonly interpret 12 as $10 + 2$, it would be more natural to partition the 4 by 12 array into two arrays in this way:



Thus, $4 \times 12 = 4 \times (10 + 2)$

$$= (4 \times 10) + (4 \times 2)$$

In order to accomplish this multiplication, it is necessary to know multiplication facts for multiples of ten. This is done for the children, also.

To find the product, 4×10, we look at

$$10 + 10 + 10 + 10 = 40 .$$

Similarly, all multiples of ten are considered by adding or counting tens. Furthermore, to multiply 3 and 20 then can be thought of as:

$$
\begin{aligned}
4 \times 20 &= 4 \times (10 + 10) \\
&= (4 \times 10) + (4 \times 10) \\
&= 40 + 40 \\
&= 80
\end{aligned}
$$

or as

$$
\begin{aligned}
4 \times 20 &= 4 \times (2 \times 10) \\
&= (4 \times 2) \times 10 \\
&= 8 \times 10 \\
&= 80
\end{aligned}
$$

In the same way, multiples of tens of tens, or hundreds can be presented, and so on.

Returning to the product of 4 and 12, it can now be completed.

$$\begin{aligned} 4 \times 12 &= 4 \times (10 + 2) \\ &= (4 \times 10) + (4 \times 2) \\ &= 40 + 8 \\ &= 48 \end{aligned}$$

We often use vertical algorithms such as these to effect the same computation.

(a) $$\boxed{\begin{array}{l} \quad (10 + 2) \\ \underline{\times \qquad\quad 4} \\ \quad 40 + 8 = 48 \end{array}}$$ or (b) $$\boxed{\begin{array}{r} 10 \\ \times\ 4 \\ \hline 40 \end{array} \qquad \begin{array}{r} 2 \\ \times\ 4 \\ \hline 8 \end{array} \qquad\qquad \begin{array}{r} 40 \\ +\ 8 \\ \hline 48 \end{array}}$$ or

(c) $$\boxed{\begin{array}{rl} 12 & \\ \times\ 4 & \\ \hline 8 & (4 \times 2) \\ 40 & (4 \times 10) \\ \hline 48 & \end{array}}$$ or (d) $$\boxed{\begin{array}{rr} 1 & 2 \\ \times & 4 \\ \hline & 8 \\ 4 & \\ \hline 4 & 8 \end{array}}$$ or eventually simply (e) $$\boxed{\begin{array}{r} 12 \\ \times\ 4 \\ \hline 48 \end{array}}$$

As another example, consider the product of the numbers 3 and 28 :

$$\begin{aligned} 3 \times 28 &= 3 \times (20 + 8) \\ &= (3 \times 20) + (3 \times 8) \\ &= 60 + 24 \\ &= 84 \end{aligned}$$

PROBLEM*

1. Show the multiplication of 3 and 28 in more detailed form, particularly in going from 3×20 to 60 and in going from $60 + 24$ to 84 .

We also may use one vertical algorithm or another to record our thinking when multiplying 3 and 28 :

---

*Solutions for problems in this chapter are on page 280.

(a) $$\boxed{\begin{array}{l} \ \ (20 + \ 8) \\ \underline{\times \qquad\quad 3} \\ \ \ 60 + 24 \\ \ \ 80 + \ 4 = 84 \end{array}}$$ or (b) $$\boxed{\begin{array}{rrr} 20 & 8 & 60 \\ \underline{\times 3} & \underline{\times 3} & \underline{+ 24} \\ 60 & 24 & 84 \end{array}}$$ or

(c) $$\boxed{\begin{array}{rl} 28 & \\ \underline{\times 3} & \\ 24 & (3 \times 8) \\ \underline{60} & (3 \times 20) \\ 84 & \end{array}}$$ or (d) $$\boxed{\begin{array}{r} 28 \\ \underline{\times 3} \\ 24 \\ \underline{6} \\ 84 \end{array}}$$ or eventually simply (e) $$\boxed{\begin{array}{r} 28 \\ \underline{\times 3} \\ 84 \end{array}}$$

Now let us extend our computation to an example such as 4×236 . We shall be fairly detailed in our first illustration:

$$\begin{aligned} 4 \times 236 &= 4 \times (200 + 30 + 6) \\ &= (4 \times 200) + (4 \times 30) + (4 \times 6) \\ &= [4 \times (2 \times 100)] + [4 \times (3 \times 10)] + (4 \times 6) \\ &= [(4 \times 2) \times 100] + [(4 \times 3) \times 10] + (4 \times 6) \\ &= (8 \times 100) + (12 \times 10) + (4 \times 6) \\ &= 800 + 120 + 24 \\ &= 800 + (100 + 20) + (20 + 4) \\ &= (800 + 100) + (20 + 20) + 4 \\ &= 900 + 40 + 4 \\ &= 944 \end{aligned}$$

PROBLEM

2. Justify each step of the procedure just illustrated for the product of 4 and 236 .

We may record our thinking in several ways using vertical algorithms:

(a) $$\boxed{\begin{array}{l} \ \ (200 + \ 30 + \ 6) \\ \underline{\times \qquad\qquad\quad 4} \\ \ \ \ 800 + 120 + 24 \\ \ \ \ 900 + \ 40 + \ 4 = 944 \end{array}}$$ or (b) $$\boxed{\begin{array}{rrrr} 200 & 30 & 6 & 800 \\ \underline{\times \ 4} & \underline{\times \ 4} & \underline{\times 4} & 120 \\ 800 & 120 & 24 & \underline{+ 24} \\ & & & 944 \end{array}}$$ or

(c) $$\boxed{\begin{array}{rl} 236 & \\ \underline{\times \ 4} & \\ 24 & (4 \times 6) \\ 120 & (4 \times 30) \\ \underline{800} & (4 \times 200) \\ 944 & \end{array}}$$ or (d) $$\boxed{\begin{array}{r} 236 \\ \underline{\times \ 4} \\ 24 \\ 12 \\ \underline{8\ \ } \\ 944 \end{array}}$$ or eventually (e) $$\boxed{\begin{array}{r} 236 \\ \underline{\times \ 4} \\ 944 \end{array}}$$

In all of these different procedures considered in this section we have seen repeatedly that use is made of the distributive property of multiplication over addition. Further extensions of multiplication to computations such as 23×45 involve even greater use of this property. However, specific consideration of these extensions is beyond the scope of this chapter.

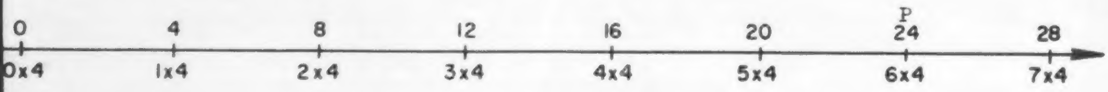
PROBLEM

3. Use one of the vertical algorithms identified above by (a) - (e) to illustrate each of these products, a. - e. respectively. For example, use (a) as a model for a.

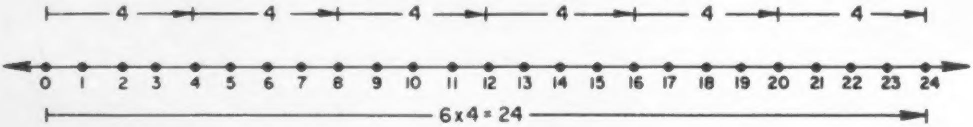
   a. 3 and 23 b. 5 and 17 c. 4 and 38
   d. 2 and 397 e. 6 and 130

DIVISION ALGORITHMS

First let us recall that a problem such as $24 \div 4 = n$ may be interpreted to mean that we are to find the number n such that $n \times 4 = 24$. We may illustrate this in the following way, using a number line representation on which we have identified multiples of 4 :

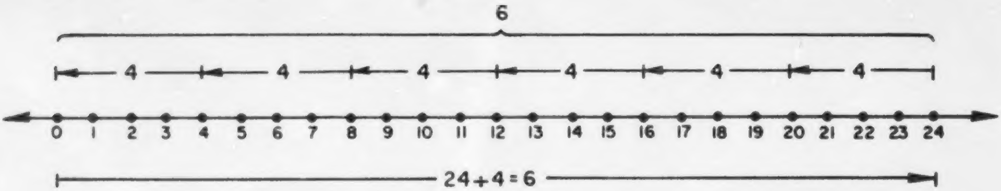


With point P we have associated 24 and also 6×4. Since the association of a number with a point is unique, we know that $6 \times 4 = 24$ and that 6 is the number n such that $n \times 4 = 24$. Let us recall what 6×4 means, using the number line. It has been interpreted in terms of repeated addition, namely $4 + 4 + 4 + 4 + 4 + 4$.



Because division is the inverse operation of multiplication and subtraction is the inverse operation of addition, it is reasonable to expect that division may be interpreted in terms of subtraction. This is indeed true.

Thus, 24 ÷ 4 can be shown on the number line as repeated subtraction.



The procedure illustrated above can be stated in terms of numbers: from 24 we subtract 4 and then continue to subtract 4 from each remainder in turn, until reaching a remainder that is less than 4 . For instance:

$$\begin{array}{r}24\\-\ 4\\\hline 20\end{array}\qquad\begin{array}{r}20\\-\ 4\\\hline 16\end{array}\qquad\begin{array}{r}16\\-\ 4\\\hline 12\end{array}\qquad\begin{array}{r}12\\-\ 4\\\hline 8\end{array}\qquad\begin{array}{r}8\\-\ 4\\\hline 4\end{array}\qquad\begin{array}{r}4\\-\ 4\\\hline 0\end{array}$$

Since there are 6 such subtractions and the resulting remainder is 0 , we know that $6 \times 4 = 24$.

Frequently we show these subtractions in a more compact form such as that shown at the right. ⟶

$$\begin{array}{r}24\\-\ 4\\\hline 20\\-\ 4\\\hline 16\\-\ 4\\\hline 12\\-\ 4\\\hline 8\\-\ 4\\\hline 4\\-\ 4\\\hline 0\end{array}$$

Our work might be shortened if, for instance, we subtracted multiples of 4 that are greater than 4 , such as:

$$\begin{array}{rl}24&\\-\ 8&(2\text{ fours})\text{ or }(2\times 4)\\\hline 16&\\-12&(3\text{ fours})\text{ or }(3\times 4)\\\hline 4&\\-\ 4&(1\text{ four})\text{ or }(1\times 4)\\\hline 0&\end{array}$$

A total of 6 fours has been subtracted since

$$\begin{aligned}(2\times 4)+(3\times 4)+(1\times 4)&=(2+3+1)\times 4\\&=6\times 4\ .\end{aligned}$$

Repeated subtraction, then, provides the rationale for division algorithms. Using multiples of the divisor can be of great advantage if we are dividing larger numbers: for example, $42 \div 3 = n$.

(a)

$$\begin{array}{rl} 42 & \\ -\underline{24} & (8 \times 3 = 24) \\ 18 & \\ -\underline{15} & (5 \times 3 = 15) \\ 3 & \\ -\underline{3} & \underline{(1 \times 3 = \ \ 3)} \\ 0 & (14 \times 3 = 42) \end{array}$$

or simply (b)

$$\begin{array}{r|r} 3\overline{)42} & \\ \underline{24} & 8 \\ 18 & \\ \underline{15} & 5 \\ 3 & \\ 3 & 1 \\ \hline 0 & 14 \end{array}$$

Thus, 24 ÷ 3 = 14 .

(a)

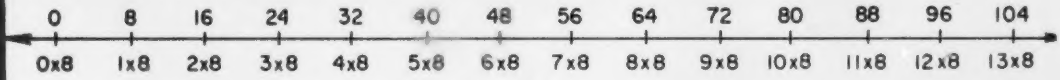
$$\begin{array}{rl} 42 & \\ -\underline{30} & (10 \times 3 = 30) \\ 12 & \\ -\underline{12} & \underline{(\ 4 \times 3 = 12)} \\ 0 & (14 \times 3 = 42) \end{array}$$

or simply (b)

$$\begin{array}{r|r} 3\overline{)42} & \\ \underline{30} & 10 \\ 12 & \\ 12 & 4 \\ \hline 0 & 14 \end{array}$$

As before, of course, 42 ÷ 3 = 14 , even though different multiples of 3 were used. Choosing multiples of ten may again be more natural and more simple eventually. However, children will begin with the smaller multiples and take larger jumps in accordance with their maturity.

Next let us consider an example such as 101 ÷ 8 = n .



Clearly there is no whole number n such that n × 8 = 101 , since 12 × 8 = 96 and 13 × 8 = 104 , and there is no whole number between 12 and 13 .

Let us explore the situation further in this way:

(a)

$$\begin{array}{rl} 101 & \\ -\underline{80} & (10 \times 8 = 80) \\ 21 & \\ -\underline{16} & \underline{(\ 2 \times 8 = 16)} \\ 5 & (12 \times 8 = 96) \end{array}$$

or simply (b)

$$\begin{array}{rr} 8\overline{)101} & \\ \underline{80} & 10 \\ 21 & \\ 16 & 2 \\ \hline 5 & 12 \end{array}$$

or eventually (c)

$$\begin{array}{r} 12 \\ 8\overline{)101} \\ \underline{8} \\ 21 \\ \underline{16} \\ 5 \end{array}$$

Thus, although there is no whole number n such that n × 8 = 101 , we have determined that 101 = 96 + 5 or 101 = (12 × 8) + 5 . However, we are *not* permitted to write something such as $101 \div 8 = 12 \text{ r } 5$, since "12 r 5" *is not a name for a number*.

In general, if a is any whole number and b is any counting number, we may associate with $a \div b$ or $\frac{a}{b}$ the sentence

$$a = (n \times b) + r$$

commonly written in the form

$$\begin{array}{r} n\,\frac{r}{b} \\ b\overline{)a} \end{array}$$

for which n is a unique whole number such that $(n \times b) \leq a$ and $r < b$. For example, $20 \div 3$ can be associated with

$$20 = (6 \times 3) + 2 \quad \text{or} \quad \begin{array}{r} 6\frac{2}{3} \\ 3\overline{)20} \end{array}$$

where $a = 20$, $b = 3$, $n = 6$ and $r = 2$. In more detail, the common algorithm would appear:

$$\begin{array}{r} 6\,\frac{2}{3} \\ 3\overline{)20} \\ \underline{18} \\ 2 \end{array}$$

Thus, $6 \times 3 = 18$ is to be subtracted from 20 to find the remainder. In order to subtract, then, 18 must be less than or equal to 20. If the remainder, 2, had been greater than or equal to 3, we could have found a larger multiple of 3 to subtract from 20.

The condition that $r < b$ has a further implication. It is certainly true that $20 \div 3$ can be associated with this equation:

$$20 = (1 \times 3) + 17$$

from which it can be stated that $20 \div 3$ is 1 with a remainder of 17. Similarly,

$$\begin{aligned} 20 &= (2 \times 3) + 14 \\ 20 &= (3 \times 3) + 11 \\ 20 &= (4 \times 3) + 8 \\ 20 &= (5 \times 3) + 5 \\ 20 &= (6 \times 3) + 2 \end{aligned}$$

are all valid equations associated with $20 \div 3$. It is generally understood, however, that when we wish to know what 20 divided by 3 is, we want the quotient expressed as the largest possible whole number plus a nonnegative remainder. (Note that there is always a remainder. When

b is a factor of a, it happens to be 0.) Thus by restricting the remainder, r, to be less than the divisor, b, we assure that n will be the largest whole number of times b is contained in a, and so we only associate with $20 \div 3$ the equation we want:

$$20 = (6 \times 3) + 2 .$$

Now let us use division algorithms to find n and r for this expression: $250 \div 7$ or $\frac{250}{7}$.

(a)

$$\begin{array}{rrl} & 250 & \\ - & \underline{140} & (20 \times 7) \\ & 110 & \\ - & \underline{70} & (10 \times 7) \\ & 40 & \\ - & \underline{28} & (\ 4 \times 7) \\ & 12 & \\ - & \underline{7} & \underline{(\ 1 \times 7)} \\ & 5 & (35 \times 7) \end{array}$$

or (b)

$$\begin{array}{r|r} 7\overline{)250} & \\ \underline{140} & 20 \\ 110 & \\ \underline{70} & 10 \\ 40 & \\ \underline{28} & 4 \\ 12 & \\ 7 & 1 \\ \hline 5 & 35 \end{array}$$

or, using larger multiples:

$$\begin{array}{r|r} 7\overline{)250} & \\ \underline{210} & 30 \\ 40 & \\ 35 & 5 \\ \hline 5 & 35 \end{array}$$

or eventually (c)

$$\begin{array}{r} 35 \\ 7\overline{)250} \\ \underline{21} \\ 40 \\ \underline{35} \\ 5 \end{array}$$

Thus, for $a = 250$ and $b = 7$, we see that $n = 35$ and $r = 5$.

We therefore may associate with $250 \div 7$ or $\frac{250}{7}$ the sentence

$$250 = (35 \times 7) + 5 .$$

PROBLEMS

4. For each of the following write an equation of the form $a = (n \times b) + r$, such that $(n \times b) \leq a$ and $r < b$.

   a. $38 \div 5$ b. $79 \div 3$ c. $112 \div 4$

   d. $\frac{57}{6}$ e. $\frac{83}{3}$ f. $\frac{106}{2}$

5. Rewrite the general equation for the special case where $r = 0$.

Consider the example $74 \div 3 = n$, or $\frac{74}{3} = n$:

$$\begin{array}{r|r}
3\overline{)74} & \\
\underline{60} & 20 \\
\underline{14} & \\
\underline{12} & 4 \\
\hline
2 & 24
\end{array}$$

This algorithm provides us with a great deal of information.

First, since the remainder is not zero, we know that there is no whole number n such that $3 \times n = 74$. That is, 3 is not a factor of 74 .

Second, the algorithm gives us the information we need to replace n and r in the equation $74 = (n \times 3) + r$ so that $(n \times 3) \leq 74$ and $r < 3$. We now may write

$$74 = (24 \times 3) + 2 .$$

Third, although there is no whole number n such that $3 \times n = 74$, there very definitely is a rational number n such that $3 \times n = 74$. One name for that rational number is $\frac{74}{3}$, since $3 \times \frac{74}{3} = 74$. The algorithm gives us the information needed to name this rational number in a different way, in mixed form. From our knowledge of rational numbers we know that 2 (the remainder) is $\frac{2}{3}$ of 3 (the divisor); that is, $2 = \frac{2}{3} \times 3$. We then may assert that

$$74 \div 3 = 24 + \frac{2}{3} = 24\,\frac{2}{3} \quad \text{or} \quad \frac{74}{3} = 24 + \frac{2}{3} = 24\,\frac{2}{3} .$$

Thus, we know that

$$3 \times (24 + \frac{2}{3}) = 3 \times 24\,\frac{2}{3} = 74 .$$

Divisions with larger numbers follow the same ideas we have developed but are beyond the scope of this chapter.

PROBLEM

6. For each exercise of Problem 4, express the quotient as a rational number in mixed form or as a whole number.

SUMMARY

In the development of multiplication algorithms we used extensively the distributive property of multiplication over addition, coupled with the renaming of a factor in accord with our decimal place value numeration scheme. For instance, in order to effect the product of 4 and 23, we renamed 23 as $(20 + 3)$ and then applied the distributive property:

$$4 \times (20 + 3) = (4 \times 20) + (4 \times 3) .$$

In the development of division algorithms we utilized a process of "repeated subtraction" in which we successively subtracted multiples of the divisor. We saw that the greater the size of the multiples used, the more efficient is the algorithm.

The division algorithm gives the information necessary to associate with $a \div b$ or $\frac{a}{b}$ (where a is any whole number and b is any counting number) either of two things:

1. an equation of the form $a = (n \times b) + r$, where $(n \times b) \leq a$ and $r < b$,
2. a rational number in mixed form whenever $a > b$ and b is not a factor of a.

A special case of both 1. and 2. arises when $r = 0$; that is, when b is a factor of a.

APPLICATIONS TO TEACHING

It is important that algorithms are developed from the standpoint of being written records of thinking patterns used when computing. Thus, we can expect that children's algorithms will change with the passing of time. At first the multiplication and division algorithms may be more lengthy and less efficient than at a later stage of work. We should allow children to use those algorithms that are most helpful and sensible to them. We may encourage them to shorten algorithms over a period of time, but children should not be forced to use more efficient algorithms prematurely.

QUESTION

"Isn't the 'platform' method terribly inefficient for division?"

To an adult who already knows how to compact his computational techniques, it may be immediately apparent that $275 \div 7 = 39\frac{2}{7}$. For the child in the beginning stages of learning these techniques, a strategy should be provided whereby he can attack such problems piecemeal without being overwhelmed by the task. Thus, he might begin with the algorithm indicated in (a):

(a)

$$\begin{array}{r|l} 7\overline{)275} & \\ \underline{70} & 10 \\ 205 & \\ \underline{70} & 10 \\ 135 & \\ \underline{70} & 10 \\ 65 & \\ \underline{35} & 5 \\ 30 & \\ \underline{21} & 3 \\ 9 & \\ 7 & 1 \\ \hline 2 & 39 \end{array}$$

(b)

$$\begin{array}{r|l} 7\overline{)275} & \\ \underline{210} & 30 \\ 65 & \\ 63 & 9 \\ \hline 2 & 39 \end{array}$$

(c)

$$\begin{array}{r} 39\frac{2}{7} \\ 7\overline{)275} \end{array}$$

Later, the student may learn to reason with himself that

$7 \times 1 = 7,\quad$ so $\quad 7 \times 10 = 70;\quad 70 < 275$
$7 \times 2 = 14,\quad$ so $\quad 7 \times 20 = 140;\quad 140 < 275$
$7 \times 3 = 21,\quad$ so $\quad 7 \times 30 = 210; \leftarrow 210 < 275$
$7 \times 4 = 28,\quad$ so $\quad 7 \times 40 = 280;\quad 280 > 275.$

The result is, 30 is the greatest multiple of 10 that is contained in the quotient. Similar arguments bring out a refinement of (a) as that in (b). With practice, he can then be led to the usual short division form (c).

The point is, an apparently "inefficient" method allows the student to attack the problem in bite size commensurate with his maturity. Observations attest to the fact that youngsters too, will learn to recognize that they can improve on the method--especially with gentle prodding. When they do realize the inefficiency, they will shift to various refinements at rates that vary from one individual to another. Ultimately, some may get to the point where they may see right away that 275 is 5 less than 280, so $275 \div 7$ is $\frac{5}{7}$ short of 40 ($= \frac{280}{7}$), arriving instantly at $40 - \frac{5}{7}$ or $39\frac{2}{7}$ for the answer. But we are not advocating that everyone must get to such a stage of proficiency.

Another important point for consideration is that the "platform" method does relate directly with the kind of activity associated with the introduction to division. With 275 members, removing a subset having 7 members, 268 members remain; another set of 7, leaves 261, ... Obviously, this is inefficient, and we may turn to removing ten subsets at a time, leading us to the method described in (a).

VOCABULARY

Algorithm *
Array *
Associative Property of Multiplication *
Commutative Property of Multiplication *
Distributive Property of Multiplication over Addition *
Division Algorithm
Remainder

EXERCISES - CHAPTER 15

1. Use several different algorithms to compute each of these:
   a. 7×34
   b. 6×48
   c. 9×28
   d. 8×54

2. Associate two things with each of the following: an equation of the form $a = (n \times b) + r$ where $(n \times b) \leq a$ and $r < b$; and a rational number in mixed form (or a whole number if b is a factor of a).
   a. $38 \div 6$
   b. $99 \div 4$
   c. $125 \div 8$
   d. $84 \div 3$

3. a. Using the common division algorithm, find the quotient $342 \div 7$.

   b. Relate this algorithm to the more primitive algorithms used by the children when they are first introduced to division.

4. In $a = (n \times b) + r$, explain why $n \times b \leq a$ and $r < b$.

SOLUTIONS FOR PROBLEMS

1.
$$\begin{aligned} 3 \times 28 &= 3 \times (20 + 8) \\ &= (3 \times 20) + (3 \times 8) \\ &= (3 \times 2 \times 10) + (3 \times 8) \\ &= (6 \times 10) + (3 \times 8) \\ &= 60 + 24 \\ &= (60 + 20) + 4 \\ &= 80 + 4 \\ &= 84 \end{aligned}$$

2.
$4 \times 236 = 4 \times (200 + 30 + 6)$ Renaming 236

$= (4 \times 200) + (4 \times 30) + (4 \times 6)$ Distributive property of multiplication over addition

$= [4 \times (2 \times 100)] + [4 \times (3 \times 10)] + (4 \times 6)$ Renaming

$= [(4 \times 2) \times 100] + [(4 \times 3) \times 10] + (4 \times 6)$ Associative property of multiplication

$= (8 \times 100) + (12 \times 10) + (4 \times 6)$ Multiplying

$= 800 + 120 + 24$ Multiplying

$= 800 + (100 + 20) + (20 + 4)$ Renaming

$= (800 + 100) + (20 + 20) + 4$ Associative property of addition

$= 900 + 40 + 4$ Adding

$= 944$ Adding

3. a.
$$\begin{array}{r} (20 + 3) \\ \times \qquad 3 \\ \hline 60 + 9 \end{array} = 69$$

b.
$$\begin{array}{r} 10 \\ \times\ 5 \\ \hline 50 \end{array} \qquad \begin{array}{r} 7 \\ \times\ 5 \\ \hline 35 \end{array} \qquad \begin{array}{r} 50 \\ +\ 35 \\ \hline 85 \end{array}$$

c.
$$\begin{array}{r} 38 \\ \times\ 4 \\ \hline 32 \\ 120 \\ \hline 152 \end{array}$$

d.
$$\begin{array}{r} 397 \\ \times \quad 2 \\ \hline 14 \\ 18 \\ 6 \\ \hline 794 \end{array}$$

e.
$$\begin{array}{r} 130 \\ \times\ 6 \\ \hline 780 \end{array}$$

4. a. $38 = (7 \times 5) + 3$ b. $79 = (26 \times 3) + 1$ c. $112 = (28 \times 4) + 0$

d. $57 = (9 \times 6) + 3$ e. $83 = (27 \times 3) + 2$ f. $106 = (53 \times 2) + 0$

5. $a = n \times b$

6. a. $7\frac{3}{5}$ b. $26\frac{1}{3}$ c. 28 d. $9\frac{3}{6} = 9\frac{1}{2}$

e. $27\frac{2}{3}$ f. 53

Chapter 16

MEASUREMENT

INTRODUCTION

Measurement is one of the connecting links between the physical world around us and mathematics. So is counting, but in a different way. We count the number of books on the desk, but measure the length of the desk. Measurement is also a connecting link between numbers and geometric figures. To measure a line segment is to assign a number to it. This cannot be done by counting the points of the segment since there are infinitely many points in any segment. To take the place of counting the points, some new concept must be developed. The concept of "measurement" is applicable not only to line segments but in a closely related fashion to angles, areas of regions, volumes of solids, weight, time, work, energy, and many other concepts or physical entities.

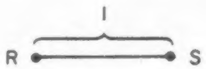
THE MEASURE OF A SEGMENT

In mathematics we think of the endpoints of a line segment as being exact locations in space. The line segment determined by these endpoints is considered to have a certain exact length. For instance, the endpoints A and B of $\overline{AB}$ are exact locations in space, and $\overline{AB}$ itself has an exact length as one of its properties. Exact length, then, is a property of all segments. In our intuitive concept of congruence, we have said that two segments are congruent if a movable copy of one can be "matched and fitted exactly" on the other. This may be interpreted as meaning that the two segments have the same length. Thus, the common property of congruent segments is the same length. Non-congruent segments have different lengths which enable them to be ordered. When we compare $\overline{AB}$ with any other segment such as $\overline{CD}$, one and only one of these three things is true:

$\overline{AB}$ IS LONGER THAN $\overline{CD}$, OR
$\overline{AB}$ IS EXACTLY AS LONG AS $\overline{CD}$, OR
$\overline{AB}$ IS SHORTER THAN $\overline{CD}$.

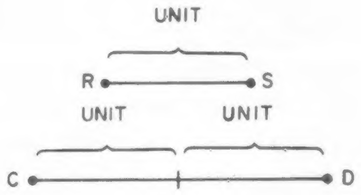
In the case of finite sets, examination revealed a property on the basis of which the sets could be compared. That is, one set could match a second set or it could have more or fewer members than the second set. At that point, numbers were associated with the property. In the same way, we wish to associate numbers with the property of length of segments. This is the objective of measurement, or finding the length of a segment.

Let us describe the process of measurement as it applies to line segments. The first step is to choose a line segment, say $\overline{RS}$, tc serve as one unit. This means to select $\overline{RS}$ and agree to consider its measure to be exactly the number 1 .



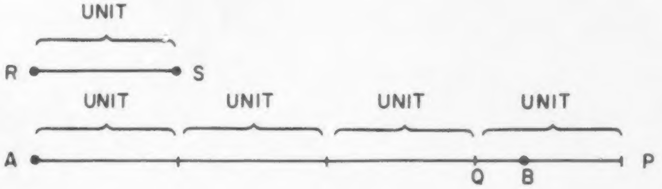
(We should recognize that this selection of a unit is an arbitrary choice we make. Different people might well choose different units and historically they have, giving rise to much confusion. For example, at one time the English "foot" was actually the length of the foot of the reigning king and the "yard" the distance from his nose to the end of his outstretched arm. Imagine the confusion when the king died if the next one was of much different stature. Various standard units will be discussed a little later but meanwhile we return to the choice of $\overline{RS}$ as our unit, recognizing the arbitrariness of this choice.)

Now it is possible to conceive of a line segment, $\overline{CD}$, such that the unit $\overline{RS}$ can be laid off exactly twice along $\overline{CD}$, as suggested in the next drawing.



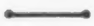
Then by agreement the measure of $\overline{CD}$ is the number 2 and the length of $\overline{CD}$ is exactly 2 units, although $\overline{CD}$ can be represented only approximately by a drawing. In the same way, line segments of length exactly 3 units, or exactly 4 units, or exactly any larger number of units are conceptually possible, although such line segments can be drawn only approximately. In fact, if a line segment is very long -- say a million inches long -- no one would want to try to draw it even approximately; but we can still think of such a segment.

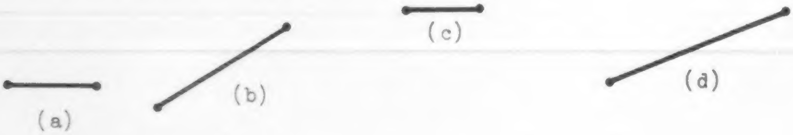
We can also conceive of a line segment, $\overline{AB}$, such that the unit $\overline{RS}$ will not "fit into" $\overline{AB}$ a whole number of times at all. $\overline{AB}$ is a line segment such that starting at A the unit $\overline{RS}$ can be laid off 3 times along $\overline{AB}$ reaching Q which is between A and B , although if it were laid off 4 times we would arrive at a point P which is well beyond B . What can be said about the length of $\overline{AB}$? Well, surely $\overline{AB}$ has length *greater than* 3 units and *less than* 4 units. In this particular case, we can also estimate visually that the length of $\overline{AB}$ is nearer to 3 units than to 4 units, so that *to the nearest unit* the length of $\overline{AB}$ is 3 units. This is the best we can do without considering fractional parts of units, or else shifting to a smaller unit.

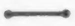


Another way of describing length to the nearest unit is by using the word "measure". Thus the *measure* of $\overline{AB}$, denoted $m(\overline{AB})$, is the number 3 . It is understood in the use of measure that it does not necessarily describe exact length. If two segments have the same length, we know they are congruent and they have the same measure. Two segments with the same measure in terms of a specified unit are not necessarily congruent. However, if two segments have the same measure for every specified unit, no matter how small, they must be congruent.

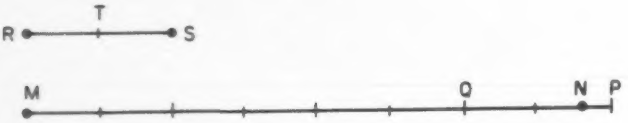
PROBLEM *

1. Using the unit  find the measure of each of the following segments to the nearest unit.



2. Using the unit  find the measure of each of the segments in Problem 1 to the nearest unit.

To help us in estimating whether the measure of a segment is say, 3 or 4 , we need to bisect our unit. $\overline{RS}$ is again shown as our unit with T bisecting $\overline{RS}$ so that $\overline{RT}$ is congruent to $\overline{TS}$ and $\overline{RS}$ is used to measure $\overline{MN}$.



In laying off the unit along $\overline{MN}$, label P the endpoint of the first unit that falls on or beyond N and label Q the end of the preceding unit just as you did for $\overline{AB}$ on the preceding page. Using $\overline{RT}$ (which has just been determined) to aid in measuring $\overline{AB}$, we can check that $\overline{BP}$ is longer than $\overline{RT}$ and that the measure of $\overline{AB}$ is 3 , or $m(\overline{AB}) = 3$. Above, $\overline{NP}$ is shorter than $\overline{RT}$ and $m(\overline{MN}) = 4$. There is nearly always a decision to be made about whether or not to count the last unit which extends beyond the endpoint of the segment being measured. The reason for this is that it is rare indeed for the unit to fit an exact number of times from endpoint to endpoint. It is

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*Solutions for problems in the chapter are on page 298.

well to realize now that measurement is approximate and subject to error. The "error" is the segment from the end of the segment being measured to the end of the last unit being counted. In $\overline{AB}$, the error is $\overline{BQ}$, in $\overline{MN}$, it is $\overline{NP}$. We note that the error in any measurement is always at most half the unit being used.

Let us emphasize one thing about terminology. In a phrase similar to "a line segment of 3 units" we mean "the measure of the line segment in terms of a particular unit is the number 3 ". The point here is simply to have a way of referring to the numbers involved so that they can be added, multiplied, etc. Remember that we have learned how to apply arithmetic operations only to numbers. You don't add yards any more than you add apples. If you have 3 apples and 2 apples, you have 5 apples altogether, because

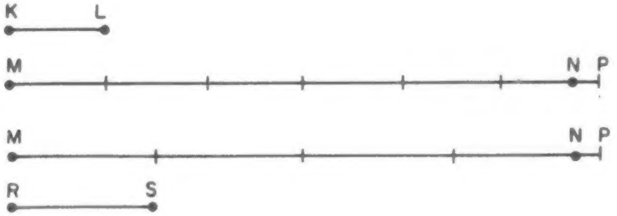
$$3 + 2 = 5 .$$

You add numbers, not yards nor apples.

As we shall see shortly, the use of different units gives rise to different measures for the same segment. Thus, if we consider $\overline{MN}$,

$m(\overline{MN}) = 6$ for the unit $\overline{KL}$ and
$m(\overline{MN}) = 4$ in terms of the unit $\overline{RS}$,

as the figure indicates.



STANDARD UNITS

Numbers of people each using their own units would have difficulty comparing their results or communicating with each other. For these reasons certain units have been agreed upon by large numbers of people and such units are called standard units.

Historically there have been many standard units used to measure line segments, such as a yard, an inch or a mile. Such a variety is a great convenience. An inch is a suitable standard unit for measuring the edge of a sheet of paper, but hardly satisfactory for finding the length of the school corridor. While a yard is a satisfactory standard for measuring the school corridor, it would not be a sensible unit for finding the distance between Chicago and Philadelphia.

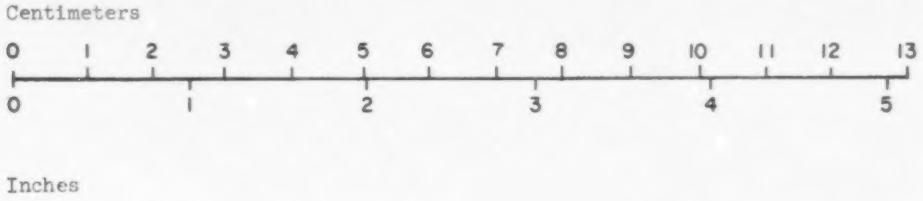
Such units of linear measure as inch, foot, yard and mile are commonly used standard units in the British-American system of measures. In the eighteenth century in France, a group of scientists developed the system of measures which is known as the metric system using a new standard unit.

In the metric system, the basic standard unit of length is the meter, which is approximately 39.37 inches or a little more than 1 yard. The metric system is in common use in all countries except those in which English is the main language spoken and is used by all scientists in the world including those in English speaking countries. Actually, the one official standard unit for linear measure even in the United States is the meter, and the correct sizes of other units such as the centimeter, inch, foot and yard are specified by law with reference to the meter.

The principal advantage of the metric system over the British-American system lies in the fact that the metric system has been designed for ease of conversion between the various metric units by exploiting the decimal system of numeration. Instead of having 12 inches to the foot, 3 feet to the yard and 1760 yards to the mile, the metric system has 10 millimeters to a centimeter, 10 centimeters to a decimeter, and 10 decimeters to a meter. This makes conversions between units very easy.

So far we have said nothing about metric units larger than the meter. The most useful of these is the kilometer, which is defined to be 1,000 meters. The kilometer is the metric unit which closely corresponds to the British-American mile. It turns out that one kilometer is a little more than six-tenths of a mile.

We have already noted that in the metric system, the meter is the unit which corresponds approximately to the yard in the British-American system. The metric unit which corresponds to the inch is the centimeter which is one-hundredth of a meter. A meter is almost 40 inches so it takes about $2\frac{1}{2}$ centimeters to make an inch or to put it another way a centimeter is about $\frac{2}{5}$ or .4 of an inch. Below are illustrated a scale of inches and a scale of centimeters so you can compare them.



SCALES AND RULERS

Once a standard unit such as a yard, meter or mile is agreed upon, the creation of a scale greatly simplifies measurement.

> A SCALE IS A NUMBER LINE WITH THE SEGMENT FROM 0 TO 1 CONGRUENT TO THE UNIT BEING USED.

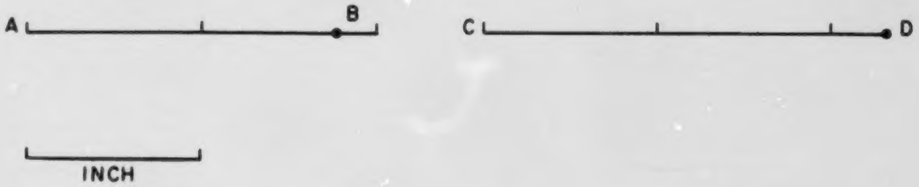
A scale can be made with a non-standard unit or with a standard unit.

> A RULER IS A STRAIGHT EDGE ON WHICH A SCALE USING A STANDARD UNIT HAS BEEN MARKED.

If we use the inch as the unit in making a ruler, we have a measuring device designed to give us readings to the nearest inch. Most ordinary rulers are marked with the unit one-sixteenth of an inch or with the unit one millimeter.

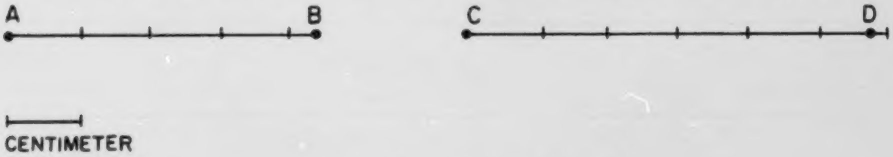
THE APPROXIMATE NATURE OF MEASURE

Any measurement of the length of a segment made with a ruler is, at best, approximate. When a segment is to be measured, a scale based on a unit appropriate to the purpose of the measurement is selected. The unit is the segment with endpoints at two consecutive scale divisions of the ruler. The scale is placed on the segment with the zero-point of the scale on one endpoint of the segment. The number which corresponds to the division point of the scale nearest the other endpoint of the segment is the measure of the segment. Thus, *every measurement is made to the nearest unit*. If the inch is the unit of measure for our ruler, then we have a situation in which two line segments, apparently not the same length, may have the same measure, in terms of a specified unit.



In inches, $m(\overline{CD}) = m(\overline{AB}) = 2$

For the same two segments we may get a different measure if we use a different unit segment. It should be clear that if the unit is changed, the scale changes. Thus, if we decide to use the centimeter as our unit, the figure below shows that in centimeters $m(\overline{AB}) = 4$ and $m(\overline{CD}) = 6$. Now the measures do indicate that there is a difference in the lengths of the two segments. Notice that by using a smaller unit (the centimeter) we are able to distinguish between the lengths of two non-congruent segments which in terms of a larger unit (the inch) have the same measure. If measurements of the same segment are made in terms of different units, the error in the measurements may be different since it is at most half the unit being used. Thus, if a segment is measured in inches the error cannot be more than half an inch, while if it is measured in tenths of an inch the error cannot be more than half of a tenth of an inch. As a result, if greater precision is desired in any measurement, a smaller unit should be used.



In centimeters, $m(\overline{CD}) > m(\overline{AB})$

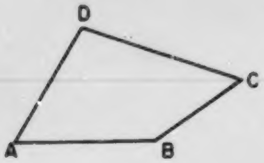
Sometimes it is more convenient to record a length of 31 inches as 2 feet 7 inches. Whenever a length is recorded using more than one unit, it is understood that the accuracy of the measure is indicated by the smallest unit named. A length of 4 yd. 2 ft. 3 in. is measured to the nearest inch. That is, it is closer to 4 yd. 2 ft. 3 in. than it is to either 4 yd. 2 ft. 2 in. or 4 yd. 2 ft. 4 in. A length of 4 yd. 2 ft. is interpreted to mean a length closer to 4 yd. 2 ft. than to 4 yd. 1 ft. or 4 yd. 3 ft. However, if this segment were measured to the nearest inch we would have to indicate this by 4 yd. 2 ft. 0 in. or 4 yd. 2 ft. (to the nearest inch). There is a very real difference in the precision of these measurements. When the measurement is made to the nearest foot, the interval within which the length may vary is one foot; when the measurement is made to the nearest inch, the interval within which the length may vary is one inch. This is because the end of the last unit counted may lie up to a half a unit on either side of the end of the segment.

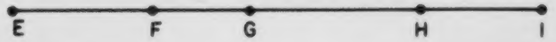
A very important property of line segments is that any line segment may be measured in terms of any given unit. This means that no matter how small the unit may be, there is a whole number n, such that if we lay off the unit n times along $\overline{AB}$ starting at A we will cover $\overline{AB}$ completely; that is, a point will be reached that is at the point B or beyond the point B on $\overline{AB}$.

The length of a line segment is a property of the line segment which we may measure in terms of different units. Theoretically, two segments have the same length if, and only if, they are congruent. We run into trouble thinking and talking about length because, in practice, measurement of length is made in terms of units and, as we saw above, two lines which are really different in length may both be said quite truly to have length 2 inches to the nearest inch.

A vivid illustration of this trouble will emerge if we think about an application of linear measurement to the calculation of the perimeter of a polygon. By definition:

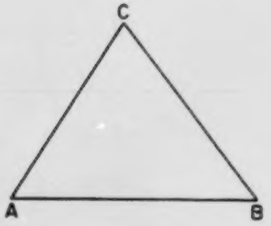
> THE PERIMETER OF A POLYGON IS THE LENGTH OF THE LINE SEGMENT WHICH IS THE UNION OF A SET OF NON-OVERLAPPING LINE SEGMENTS CONGRUENT TO THE SIDES OF THE POLYGON.

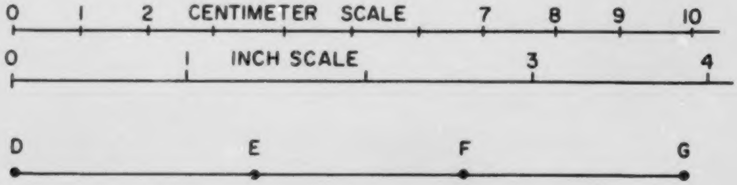




Thus the perimeter of polygon ABCD is the length of $\overline{EI}$ where $\overline{EI}$ is the union of $\overline{EF}$, $\overline{FG}$, $\overline{GH}$ and $\overline{HI}$ which are respectively congruent to $\overline{AB}$, $\overline{BC}$, $\overline{CD}$ and $\overline{DA}$. If we put pins at points A , B , C and D and stretch a taut thread around the polygon from A back to A , when we straighten out our thread we will have a model of a segment congruent to $\overline{EI}$.

The length of $\overline{EI}$, we know intuitively, is the sum of the lengths of the four segments when we consider length as an intrinsic property of segments. But, when we talk about lengths as measured in terms of certain units we may run into the following situation:





To the nearest centimeter $m(\overline{AB}) = m(\overline{BC}) = m(\overline{CA}) = 3$. $\overline{AB}$ is congruent to $\overline{DE}$, $\overline{BC}$ is congruent to $\overline{EF}$, $\overline{CA}$ is congruent to $\overline{FG}$ but $m(\overline{DG}) = 10$. This is because to the nearest millimeter $m(\overline{AB}) = m(\overline{BC}) = m(\overline{CA}) = 33$, and to the nearest millimeter $m(\overline{DG}) = 99$, and to the nearest centimeter this means $m(\overline{DG}) = 10$. Even if we measure our segments to the nearest inch we find $m(\overline{AB}) = m(\overline{BC}) = m(\overline{CA}) = 1$ and we would expect the measure of the perimeter to be 3 . But we find $m(\overline{DG}) = 4$. This reminds us again that the measure of a length is always, at best, an approximation and approximation errors may accumulate to cause real trouble. The best we can say is to be aware of this possibility whenever in your problems you are dealing with numbers which turn up from measurement processes. The preciseness of any measurement is related to the size of the unit selected.

PROBLEMS

3. Two children are asked to determine the length and width of a crate; one is given a ruler with units marked in feet, the other a ruler with units marked in inches. The first says the crate is 3 feet long and 2 feet wide; the second says it is 40 inches by 28 inches. Explain why they could both be right.

4. Both children are asked to find the perimeter of the crate. The first one says 10 feet, the second says 136 inches. A string is then passed around the crate, stretched out and the children are asked to measure the string to find the perimeter. This time the first one says 11 feet, the second one 137 inches. Which results are correct? Explain the discrepancy between the results.

We have indicated in this development, that length is the common property possessed by segments that are congruent in much the same way that a number is the common property of all sets that are equivalent. Corresponding to the length of a given segment, a whole number is attached which we call its measure. Note that this measure depends on the unit selected, and, as we have seen, is what one normally considers the measure to the nearest unit. Thus, length is approximated by the measure, with the approximation being closer and closer as the unit is finer and finer. This is the case for any measure whether it describes length, time, weight, or any other measurement.

When we say that a segment has a measurement of $3\frac{1}{4}$ inches, for instance, the implication is that the unit is the quarter-inch. Thus, a "measure" of $3\frac{1}{4}$ is actually 13, since $3\frac{1}{4}$ inches means 13 quarter-inches. When a measure is expressed as a rational number, the understanding is, therefore, that an approximation is made to the smallest unit indicated, as for example, the quarter-inch mentioned above. Starting with the concept of measure as a whole number, a meaning may now be attached to a measure given in terms of a rational number. With reference to the smaller unit, the measure is the whole number of the smaller units; with reference to the larger unit, the measure may be stated as a rational number.

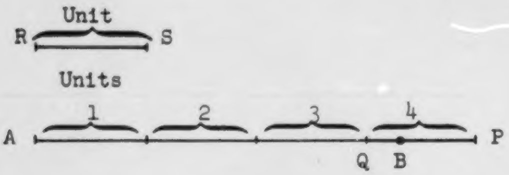
On a line, a segment can always be found that would be congruent to some segment. It is then possible to choose two points on a line so that the segment determined by the two points would be congruent to the unit for a particular measure. If the two points on the line were identified as 0 and 1 , then a number line may be constructed such that the unit on the number line is congruent to the unit for the measure.

Now, suppose that the length of a given segment is to be determined. Clearly, there would be a segment on the number line from 0 to a point having a rational number as its coordinate that would approximate the given segment in length. In fact, by finding the segment on the number line with 0 as one of the endpoints (the left endpoint) that is congruent to the segment being measured, it should be possible to obtain the measure by the coordinate of the other endpoint. By this, any number that may be associated with any point on the number line as its coordinate may be assigned as the measure of a segment, and two segments are said to be of the same length if they have the same measure regardless of the unit used. Length, conceived of as the common property of congruent segments, is a slight departure from length in ordinary language usage, as for example, in stating that the length of a desk is 4 feet. The explanation of length as the common property of congruent segments more accurately emphasizes its mathematical meaning.

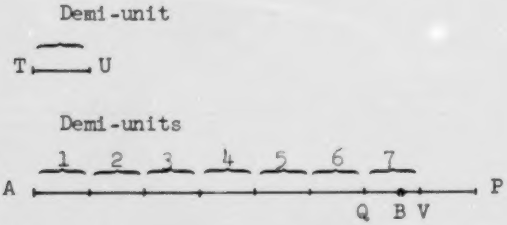
QUESTION

"Why is it said that a measure is a whole number of units when clearly we can read off measurements on the ruler such as $3\frac{1}{2}$ inches?"

The point is made in the section on the use of an arbitrary unit (page 272) that if $\overline{RS}$ is taken to be a model for the unit, then $\overline{AB}$ is between 3 and 4 units in length.



In terms of this unit, we could only say that the measure of $\overline{AB}$ is 3 since B is nearer to Q than to P, where Q is the point exactly 3 units from A and P is 4 units from A. Suppose we were to measure $\overline{AB}$ using a segment $\overline{TU}$ exactly half as long as $\overline{RS}$ to be the model for the unit. Let's call this a "demi-unit" to indicate that this is smaller than the unit determined by $\overline{RS}$. The situation may be as diagrammed below.



Since B is between Q and V, then the measure of $\overline{AB}$ is between 6 and 7. Since B is nearer to V than to Q, to the nearest unit, the measure of $\overline{AB}$ is 7. This is in terms of the demi-unit. If the demi-unit is compared to the unit as determined by $\overline{RS}$, we might be justified in thinking of it as a "half-unit". The 7 demi-units are then thought of as seven "$\frac{1}{2}$ - units" or $\frac{7}{2}$ units. For this reason, behind the statement that an object is $3\frac{1}{2}$ units in length is the concept that the "half-unit" is being used as a unit and that to the nearest one of the marked intervals, 7 is the whole number for the measure. The fact that $3\frac{1}{2}$ may be read directly from the ruler shows that we may adapt ourselves to estimating quickly and filling in subinterval marks if necessary. For example, the ruler may be marked off in thirty-seconds of an inch and we may be estimating to sixty-fourths of an inch. When we do so, we are in reality relating the whole number of units using the $\frac{1}{64}$ - inch segment as a model; furthermore, in the reading, this whole number of subunits is related back to the standard unit of an inch.

VOCABULARY

Centimeter

Exact Length

Kilometer

Length*

Line Segment*

Linear Scale*

Measure*

Meter

Metric System

Perimeter (of a polygon)*

Preciseness

Ruler*

Scales*

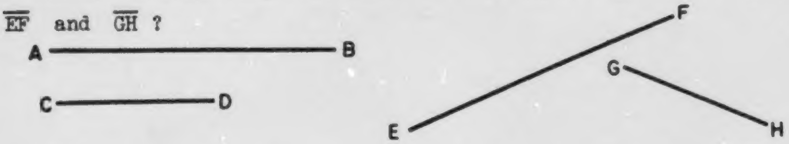
Segment*

Standard Units*

Unit*

EXERCISES - CHAPTER 16

1. Which of the following statements is true about segments $\overline{AB}$, $\overline{CD}$, $\overline{EF}$ and $\overline{GH}$?



a. $\overline{AB}$ is congruent to $\overline{CD}$

b. $\overline{AB}$ is shorter than $\overline{CD}$

c. $\overline{AB}$ is longer than $\overline{EF}$

d. $\overline{AB}$ is congruent to $\overline{EF}$

e. $\overline{GH}$ is shorter than $\overline{CD}$

f. $\overline{GH}$ is congruent to $\overline{CD}$

2. A dog weighs 18 pounds.

a. The unit of measure is __________.

b. The measure is __________.

c. The weight is __________.

3. A desk is 9 chalk pieces long.

a. Its measurement is __________.

b. Its measure is __________.

c. The unit of measure is __________.

4. In which of the following sentences are standard units used?

a. He is strong as an ox.

b. Put in a pinch of salt.

c. We drink a gallon of milk per day.

d. The corn is knee high.

e. I am five feet tall.

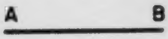
5. The measures of the sides of a triangle in inch units are 17 , 15 and 13 .

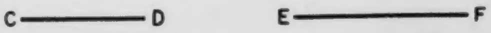
a. What are the measures of the sides if the unit is a foot?

b. What is the measure of the perimeter in inches? In feet?

c. Is there anything curious about your answer?

d. How do you explain it?

6. Use  as a unit to measure the following segments.



Is $\overline{CD}$ congruent to $\overline{EF}$? Do your answers contradict each other? Explain.

SOLUTIONS FOR PROBLEMS

1. a. 1 ; b. 2 ; c. 1 ; d. 2 .

2. a. 2 ; b. 3 ; c. 1 ; d. 3 . It should be noted how the measures differ.

3. 40 inches to the nearest foot is 3 feet since the error is less than $\frac{1}{2}$ foot. 28 inches to the nearest foot is 2 feet. Again the error is less than $\frac{1}{2}$ foot.

4. This problem involves the definition of perimeter of a polygon. Note that the perimeter is by definition the length of the segment which is congruent to the union of non-overlapping segments congruent to the sides. Thus the second method is the correct one for both children and the answers to the nearest unit are 11 feet and 137 inches. The first result comes from adding 3 + 2 + 3 + 2 but each measure had an error of about 4 inches or $\frac{1}{3}$ of a foot and the accumulation of these leads to the result 10 feet which is, in fact, incorrect. The result 136 inches comes likewise because each side measured in inches had an error less than $\frac{1}{2}$ an inch but which accumulated to something near an inch. The difference between the correct results 11 feet and 137 inches is due to the fact that each child gives his answer correct to the nearest unit he is using.

Chapter 17

STRUCTURE

THE COUNTING NUMBERS

In our development, we have started with sets as pre-number concepts and obtained from them the set of counting (natural) numbers. Although we did not consider the properties of the counting numbers (we considered properties of whole numbers), if we had examined the counting numbers in this light, we would have discovered closure under addition and multiplication. In fact, all of the properties listed below hold for the set of counting numbers:

1. the set is closed under addition and multiplication;
2. the elements are commutative under addition and multiplication;
3. the elements are associative under addition and multiplication;
4. there is an identity element for multiplication;
5. multiplication is distributive over addition.

The statement for the closure property under addition is: if a and b are counting numbers, then $a + b$ is a counting number. This may also be stated:

IF a AND b ARE COUNTING NUMBERS, AND
$a \times b = c$, THEN c IS A COUNTING NUMBER.

Thus, if a is 3 and b is 5, then c is $3 + 5$, or 8. A related question is: if a is 3 and c is 8, is there a counting number x such that $a + x = c$? In terms of open sentences, we are then looking for the solution for

$$3 + x = 8 .$$

In this case, 5 is the solution of the equation. If we ask whether there is a counting number b such that $3 + b = 8$, we are posing the question: Is $3 + b = 8$ solvable in the set of counting numbers?

THE WHOLE NUMBERS

In our study, we have found that $3 + 0 = 3$; furthermore, 0 is the only solution for $3 + x = 3$. However, 0 is not a counting number.

Clearly then, $3 + x = 3$ is not solvable in the set of counting numbers. Nor are $5 + x = 5$, $6 + x = 6$, $2 + x = 2$, and so on. In fact, for any counting number a , 0 is the only solution for

$$a + x = a ,$$

and hence, $a + x = a$ has no solution in the set of counting numbers.

By adjoining 0 to the set of counting numbers, we obtain an extension from the counting numbers to the whole numbers. That is,

$$\text{IF } Z = \{0\} \text{ AND } N = \{1 , 2 , 3 , 4 , 5 , \dots\} ,$$
$$\text{THEN } Z \cup N = \{0 , 1 , 2 , 3 , 4 , 5 , \dots\} = W .$$

Within the set of whole numbers, then, the equation $a + x = a$ has the solution $x = 0$. All the properties that we have for the set of counting numbers hold equally for the set of whole numbers. By the inclusion of 0 in the set of whole numbers some new properties are gained:

THERE IS AN IDENTITY ELEMENT FOR ADDITION;
THE PRODUCT OF 0 AND ANY WHOLE NUMBER IS 0.

THE INTEGERS

Even adjoining 0 to the set of counting numbers is not enough to completely solve the equation, $a + x = c$. If $c < a$, this equation is not solvable in the set of whole numbers. For example, there is no whole number x such that $5 + x = 3$. Negative numbers are introduced in the first grade, but only in a limited way in relation to the number line, for example, as associated with the scale on a thermometer. Later on, when negative numbers are explored in greater detail, the opposites of the counting numbers, namely, $\{\dots , {}^-4 , {}^-3 , {}^-2 , {}^-1\}$, may be adjoined to the whole numbers. Thus, we get the set of integers

$$I = \{\dots , {}^-4 , {}^-3 , {}^-2 , {}^-1 , 0 , 1 , 2 , 3 , \dots\} .$$

Then, the equation $a + x = c$ will be solvable in the set of integers for numbers a and c in this set. By this extension, we will find that all the properties that we have identified for the whole numbers still hold for the integers. Moreover, we have an additional property which derives from the solvability of $a + x = 0$ for any integer a . The solution for this equation is called the inverse of a . The property may be stated:

> FOR EACH INTEGER a, THERE IS AN INVERSE,
> ^-a SUCH THAT $a + {}^-a = 0$.

By the commutative property, we can see that a and ^-a are inverses of each other. For example, $3 + {}^-3 = 0$ and $^-3 + 3 = 0$; so 3 and $^-3$ are inverses of each other.

Historically, there was only need of the counting numbers for the primitive man; his possessions and all his reckoning were adequately accounted for by these numbers. The concept of zero as a number did not emerge until quite late in civilization. With sophistication, we may interpret the concept from a different point of view. Zero might be considered to be the solution for $a + x = a$ for whatever number a ; in this way, a number called zero is "postulated" as the solution. Similarly, ^-a may be postulated as the solution for $a + x = 0$.

THE RATIONAL NUMBERS

We may next consider the solvability of equations of the form $a \times x = c$ for integers a and c . Evidently, for certain numbers such as $a = 2$ and $c = 6$, the equation $a \times x = c$ is solvable in integers. The solution for $2 \times x = 6$ is 3 . However, equations such as

$$6 \times x = 2 ,$$

are not solvable in the set of integers. This leads to the set of all rational numbers: numbers represented by $\frac{m}{n}$ where m and n are integers and $n \neq 0$. The solution for $6 \times x = 2$ is then considered to be $\frac{2}{6}$ just as the solution for $2 \times x = 6$ is considered to be $\frac{6}{2}$. As we have indicated in the preceding section regarding the postulation of zero and ^-a , the number $\frac{m}{n}$ may also be postulated as the solution for $n \times x = m$.

By representation of such numbers on the number line, we identified, for example, the numbers named as

$$\frac{3}{1},\ \frac{6}{2},\ \frac{9}{3},\ \ldots,\ \frac{m \times k}{n \times k},\ \ldots,\ \text{for } k \neq 0$$

to be the same number. Thus,

> if a and b are nonnegative integers such that $b \times k \neq 0$, then all numbers that can be represented by $\frac{a \times k}{b \times k}$ are identified with $\frac{a}{b}$ and all numbers that can be represented by $\frac{^-a \times k}{b \times k}$ are identified with $^-\frac{a}{b}$, where a and b do not have any common factor other than 1 (unless $a = 0$).

In this way, $\frac{1}{3}$, $\frac{2}{6}$, $\frac{3}{9}$, ... are considered to be in the same "equivalence" class; $\frac{2}{3}$, $\frac{4}{6}$, $\frac{6}{9}$, ... in another equivalence class; $\frac{1}{2}$, $\frac{2}{4}$, $\frac{3}{6}$, ... in still another class; and so on. Corresponding to the equivalence of $\frac{2}{3}$, $\frac{4}{6}$, $\frac{6}{9}$, ... is the equivalence of the statements

$$3 \times x = 2 , \quad 6 \times x = 4 , \quad 9 \times x = 6 , \quad \ldots .$$

So, instead of defining the equivalence classes via the number line, the concept also can be approached via equivalent statements. Either way, $\frac{2}{3}$, $\frac{4}{6}$, $\frac{6}{9}$, ... would be classified together. Our approach by the number line is the more intuitive approach in accord with the presentation to the students.

There is another kind of identification that we might interpret by the number line. It is that the rational numbers $\frac{m}{1}$, $\frac{m \times 2}{1 \times 2}$, $\frac{m \times 3}{1 \times 3}$, ... , may be identified with the integer m, if m is an integer. From this viewpoint, the set of rational numbers may be regarded as an extension of the set of integers. We can observe that in the set of rationals, all the properties that we have identified that hold for the integers still hold. Furthermore, another property is gained -- one that parallels the property on inverses under addition:

> FOR EACH RATIONAL NUMBER $\frac{m}{n}$ THAT IS DIFFERENT FROM 0, THERE IS AN INVERSE $\frac{p}{q}$ SUCH THAT $\frac{m}{n} \times \frac{p}{q} = \frac{1}{1}$ (with the identification, $\frac{1}{1} = 1$).

For example, $\frac{2}{3} \times \frac{3}{2} = \frac{2 \times 3}{3 \times 2} = \frac{6}{6} = \frac{1}{1}$.

With extension on top of extension, we see an emerging structure of the numbers as characterized by the properties. Each set of numbers, together with the operations and the properties, form what is called a number system. For the rational number system, the properties may be listed as follows:

the set is closed under addition and multiplication, for example, $\frac{1}{2} + \frac{5}{3}$ is a rational number;

the elements are commutative under addition and multiplication, for example, $\frac{1}{2} + \frac{5}{3} = \frac{5}{3} + \frac{1}{2}$;

the elements are associative under addition and multiplication, for example, $(\frac{1}{2} + \frac{5}{3}) + \frac{3}{4} = \frac{1}{2} + (\frac{5}{3} + \frac{3}{4})$

there is an identity element for addition, for example, $\frac{1}{2} + 0 = \frac{1}{2}$;

there is an identity element for multiplication, for example, $\frac{3}{4} \times 1 = \frac{3}{4}$;

for each rational number, there is an inverse under addition, for example, $\frac{2}{3} + {}^{-}(\frac{2}{3}) = 0$;

for each rational number different from 0 , there is an inverse under multiplication, for example, $\frac{5}{6} \times \frac{6}{5} = 1$;

multiplication is distributive over addition, for example, $\frac{1}{2} \times (\frac{2}{3} + \frac{5}{7}) = (\frac{1}{2} \times \frac{2}{3}) + (\frac{1}{2} \times \frac{5}{7})$.

Besides these, there are properties which we can elicit from the above, such as

the product of 0 and any rational number is 0 ; for example, $0 \times \frac{9}{7} = 0$.

OTHER EXTENSIONS

Other extensions will be made beyond the set of rational numbers, but these will not be carried out in the first six grades. The rational numbers were associated with points on the number line. As the rational numbers have the property of being dense (between any two rational numbers are infinitely many rational numbers), it appears that every point on the number line represents a rational number. However, there *are* numbers such as π , $\sqrt{2}$, $\sqrt[3]{7}$, and so on, that are coordinates of points on the number line but are not rational numbers.

The next extension brings us the set of all numbers that may be represented on the number line. These are the real numbers. Beyond this extension are the complex numbers, whose representations occupy the entire coordinate plane (that is, just the number line is not sufficient for their respresentations) and the hypercomplex numbers. With each number system is associated a structure given by its properties.

We have pointed to the property or properties gained with each extension. However, although we shall not show how here, we should mention that it is not always the case that properties are gained. The extension from the complex numbers to a hypercomplex system may result in the loss of the commutative property; a further extension may result in the loss of both the commutative and associative properties.

There are other losses of properties that occur in the extensions which have not been mentioned but which we will note very briefly now. When the set of whole numbers is extended to the set of integers, we lose the property that there is a number which we can call a first (or smallest) number. Extending to the rationals, we lose the property that each number has a number which we call the *next* number (or successor). That is, the integers can be visualized as "isolated" (discrete) points on the number line, whereas the rationals are visualized as being densely packed. It can be shown that the rational numbers may be put into 1-1 correspondence with the counting numbers, whereas a 1-1 correspondence cannot be made with the real numbers (we say that we lose the property of countability in the extension). The extension from the real numbers to the complex numbers results in loss of the property of order: between two complex numbers, there is no "order relation" such as "<" or ">" that determines which of the two numbers precedes the other.

While we have losses with the extensions mentioned, the gains apparently far outweigh the losses, considering the many, many new problems that can be solved with each extension. An important aspect in the study of algebraic extensions consists of determining properties that hold in each extension. In turn, the study may orient itself to investigating what extensions may be determined that would retain certain properties (such as associativity, etc.), and this is indeed a program in the study of algebra.

An appropriate observation to make at this time is that in presenting mathematics as a structured discipline, the student is guided through the extensions of the number systems. Thus, with the student's maturity, his knowledge of systems of numbers is simultaneously broadened and deepened.

VOCABULARY

Associative Property*

Closure Property*

Commutative Property*

Complex Numbers

Counting Numbers*

Distributive Property of Multiplication over Addition*

Identity Element*

Integers*

Inverses under Addition*

Inverses under Multiplication*

Rational Numbers*

Real Numbers

Whole Numbers*

APPENDIX A

THE MATHEMATICS PROGRAM, GRADES K-3

The SMSG mathematics program, MATHEMATICS FOR THE ELEMENTARY SCHOOL, K-3, is a contemporary instructional program that emphasizes conceptual learning. Primary attention is given to the introduction and progressive development of significant mathematical ideas. This emphasis on mathematical ideas provides the necessary foundation for the related development of appropriate skills and the ability to use mathematics effectively.

Central to the program are relatively few basic ideas. Two of these are the ideas of number and operation. Each is introduced and extended in close association with appropriate manipulations of sets of physical objects. Major attention is given to the set of whole numbers, $\{0, 1, 2, 3, 4, \ldots\}$, and to the nature and properties of the familiar operations of addition, subtraction, multiplication, and division within the set of whole numbers. Consideration is given also to the nonnegative rational rational numbers, such as $\frac{1}{2}$, $\frac{2}{3}$, $\frac{3}{4}$, $\frac{5}{6}$, $\frac{7}{8}$.

Closely related to the ideas of number and operation is the idea of numeration and also the ability to compute. Emphasis is given to the decimal base (ten) of our numeration system and to its "place value" principle. These, coupled with properties of the operations, form the basis for developing meaningful algorithms--i.e., forms for computing.

The remaining major idea developed in the program is geometric in nature. Our first concern is with characteristics and properties of familiar geometric figures, viewed as sets of points and abstracted from appropriate models within the physical world. A related concern is with measurement, in which number is applied to properties such as length and area of geometric figures.

The Teachers' Commentary for Book K stresses informal work with pre-number concepts at the kindergarten level. These are concepts pertaining to sets, and are explored through work with sets of physical objects. The ideas of number and geometry also are introduced at the kindergarten level, as reflected in the scope and organization chart. Not all topics in the student texts, Books 1-3 are included in this chart. However those that are listed are tied to the topics in this inservice text.

SCOPE AND ORGANIZATION OF MATHEMATICAL CONTENT

For expediency, a notation such as 2.3 has been used in this chart to refer to the chapter and section numbers in Books 1-3. The notation, 2.3, means Chapter II, Section 3. in the particular text.

Topic	K	1	2	3
Sets Members of a set	1	1.1	1.1	
The empty set	1	1.2	1.1	
Number of members	1	1.5, 1.7	1.1	
Pairing and equivalence	3	1.3		
Comparison of sets	3	1.4	1.2	2.2
Ordering of sets	7	2.1	1.2	2.2
Joining sets	5	4.1,4.2, 4.3	1.3	2.1
Subsets Removing subsets and the remaining set	5	4.4,4.5, 4.6	1.4	2.1
Sets of points Point			3.1	1.1
Curve	2	5.2	3.2	1.1
segment		10.1	3.2	1.1
ray			7.2	1.2
line			3.3	1.2
simple closed curve	2	5.3	3.4	1.3
circle	2	5.2		
rectangle	2	5.3		1.3
triangle	2	5.3	3.5,7.1	1.3,1.5
quadrilateral		5.3		1.3
square	2	5.3		1.3
pentagon				
hexagon				
interior, exterior, on	2		3.5	1.4
angle			7.3,7.4, 7.5	1.2,1.5
congruence		5.5	7.1-7.5	

Topic	K	1	2	3
Region	2, 8	5.4	7.1	1.4
Solid	2	5.1		
Comparison of sizes	3			
Linear measurement		10.2	5.1-5.4	6.1,6.2
length to nearest unit		10.3	5.5	6.3
A real measurement				6.5,6.7
The number line				
Coordinates		2.1,2.2, 2.3	1.8,2.1	8.1,3.2
Place value				
Sets of ten		3.1	6.1	2.4
The written numerals through 99		6.1-6.5		
Operation on whole numbers				
Addition				
by joining		4.3		
using the number line		7.3	2.1	
Subtraction				
by removing subsets		4.5		
using the number line		7.3		
by missing addend		7.5		
Multiplication				
arrays		8.1	8.1	4.1
relation to multiplication		8.2	8.2	4.2
by repeated addition			8.4	4.2
using number line				3.5
factors prime			9.1	4.4
Division				
arrays			9.1	
relation to multiplication			9.1	
by repeated subtraction			9.3	
finding factors			9.1	
Rational numbers			9.1-9.4	
Partitioning,parts of regions		9.1-9.6		8.1
Rationals on number line			9.5	
Equivalent subsets				8.2-8.4
Order of rational numbers				8.5

Topic	K	1	2	3
Operations on rational numbers				
Addition			9.5	
Multiplication		9.6		
Techniques of computations				
Addition facts			2.3	2.3,2.5
addition algorithm			6.2-6.7	5.1-5.4
Subtraction			6.2,6.6	
subtraction facts				2.3,2.6, 5.1
Multiplication facts			8.4	4.3
multiplication algorithm				7.1
Finding quotients			9.1-9.3	7.3-7.5, 8.1-8.4
Coordinates in a plane				3.3

The program of Book 1 reviews and extends pre-number concepts associated with sets, and also reviews and extends the work with numbers and geometry. Operations and numeration are introduced, along with measurement. The extent of this work is reflected in the scope and organization chart above.

It is imperative that kindergarten and first-grade teachers view their instructional work in relation to the work of subsequent grades. All of the basic ideas of the program appear within Book 1 and are extended in Book 2 and Book 3.

This inservice book develops the mathematics underlying the program of Grades K-3. By using the chart above, the teacher will be able to see how these ideas are progressively developed from kindergarten through Grade 3.

The nature and scope of mathematical content embraced by MATHEMATICS FOR THE ELEMENTARY SCHOOL, K-3 interest and challenge children within the primary grades. They also interest and challenge the teachers of these children. Through this program mathematics truly can "come alive" for both children and teachers.

APPENDIX B

LANGUAGE AND MATHEMATICAL INSTRUCTION

INTRODUCTION

The introductory chapter, Chapter 0, included a consideration of language characteristics of culturally disadvantaged children and some general implications of these characteristics for teaching these children. In this appendix we shall deal more explicitly with language and mathematical instruction.

Mathematics is a language. It provides a precise means of communicating such ideas as number and space. It has special terms, expressions, and symbolism which, if understood, facilitate its use as a language and which, if not understood, inhibit its use as a language.

A major objective of mathematics instruction in the elementary grades is growth in children's ability to use the language of mathematics effectively. This includes growth both in understanding mathematical concepts and in knowledge of terms, expressions, and symbols associated with these concepts.

Growth in the ability to use the language of mathematics effectively and in the ability to use "general" language effectively are closely related. The development of these should go hand in hand, each reinforcing the other. In this sense, everything you do to improve the general language ability of children can reinforce the development of their mathematical language ability; conversely, everything you do to improve the mathematical language ability of children can reinforce the development of their general language ability.

Here, we shall discuss the importance of aural-oral experiences and the learning of concepts, terms, and symbols. We shall consider the preciseness of mathematical language and the use of correct or preferred terminology in teaching young children. Multiple meanings of terms as well as the importance of distinctions between terms will be illustrated by examples from classroom situations.

AURAL-ORAL EXPERIENCES

In keeping with the development of young children's general language abilities, aural-oral experiences precede and receive greater emphasis than do reading-writing experiences in the early stages of the development of mathematical abilities. These listening-speaking experiences need extensive attention.

It is important to note here several kinds of difficulties which children may have in aural-oral learning.

1. Some terms are completely new to children and, if polysyllabic, may not be heard or spoken correctly at the outset. A good illustration of this is the term *equivalent*.

2. Some expressions are relatively lengthy ones, and children may attend to only part(s) of the expression rather than to the whole expression. For instance, with the expression *as many as*, or *as many members as*, children often focus attention on just the "many" or "many members." They do not even sense the word "as" in the expressions.

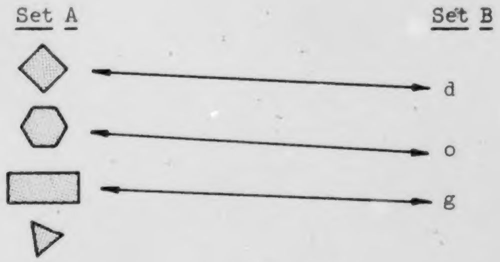
3. Some words sound so much alike that children may fail to sense any distinction between them. Examples of this are the words *six* and *sixth*, or *ray* and *array*. The latter may also be confused with *a ray*.

Difficulties such as these make it imperative that we give most careful attention to speaking and to listening experiences.

Setting Language Patterns

We wish to emphasize the "power of example" in developing acceptable mathematical language patterns of young children--i.e., the power of *your* example.

If terms are to be used correctly by the children, the terms first must be used correctly by the teacher. Consider the following.



In this instance we would *avoid* saying, "Set A is larger than set B." Rather, we would say "Set A has *more members than* set B." We would *avoid* asking, "Which set is larger?" Rather, we would ask, "Which set has more members?"

A child might say, "Set A is larger." Without making an issue of this, we can accept what he has said but respond by giving a preferred language pattern: "Yes, Tommy, set A has more members."

As we work to establish preferred language patterns, it is important that we ourselves enunciate as clearly and as distinctly as possible. This is important at all times, but it is particularly crucial when distinguishing between words such as fifteen and fifty, nineteen and ninety. Failure to enunciate clearly and distinctly can lead to such expressions as "quivun" for "equivalent."

Use of Primitive Language

Primitive language often must precede the use of the technical language of mathematics. For instance:

A rectangle is a quadrilateral whose sides determine four right angles (i.e., each pair of adjacent sides determines a right angle). We obviously cannot expect young children to use such terminology from the outset of their work with rectangles!

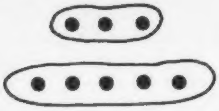
Consequently, with young children we refer simply to the "corners" of a rectangle rather than to the angles associated with a rectangle. Although children certainly could be taught to use the word "angle" from the outset, it would be a meaningless or misunderstood term at that time. Concepts of angle, right angle, and angle associated with a rectangle are much too sophisticated to be used with understanding when young children first work with rectangles. Consequently, at first we have no reasonable alternative but to use primitive terms such as "corner."

Premature use of technical mathematical language, before such terms and expressions can be understood by children, should be avoided.

CONCEPTS, TERMS, AND SYMBOLS

We must distinguish carefully between a child's understanding of a mathematical concept and his ability to use mathematical terms or symbols associated with that concept.

First consider the idea that the order in which two whole numbers are added does not effect their sum, e.g. $3 + 5 = 8$ and $5 + 3 = 8$, or $3 + 5 = 5 + 3$. We refer to this as the commutative property of addition within the set of whole numbers. Children may show in various ways an understanding of this idea. For instance, with the two sets of dots below they may associate either $3 + 5 = 8$ or $5 + 3 = 8$.



However, they may be wholly unaware that they are dealing with a particular instance of the commutative property of addition. An understanding of the concept may exist without knowledge of its technical name. Also, knowledge of the expression, *commutative property of addition*, in no way indicates or guarantees that children understand the concept.

Now consider the idea of *greater than* as it applies to whole numbers. Children may know very well that 7 is greater than 4. They may be able to demonstrate this fact by showing that a set of 7 things has more members than a set of 4 things. These same children, however, may not be able to write $7 > 4$, nor understand what $7 > 4$ means when they see it written. An understanding of the concept may exist without knowledge of symbolism associated with that concept.

Our first concern is with the development of mathematical concepts and understandings. Technical terms and symbols are introduced and used only when it becomes advantageous to do so. Frequently this comes much later than the introduction of the concept itself.

Preciseness of Language

One characteristic of mathematical language is its *preciseness*. Consider our use of the expression, *is equal to* (or, *equals*).

When we state that "set A is equal to set B" $(A = B)$, we mean simply and precisely that "A" and "B" are names for the *same* set. When we state that "$3 + 4 = 7$" we mean that "$3 + 4$" and "7" are names for the *same* number; when we state that "$6 - 1 = 5$" we mean that "$6 - 1$" and "5" are names for the *same* number. Similarly, when we write "$\overline{AC} = \overline{CA}$" we mean that "AC" and "CA" are names for for the *same* line segment.

In each of these instances, and throughout our work, the expression *is equal to* is used to convey precisely the same meaning. We have asserted that *one* thing--a set, a number, etc.,--has been named in two ways. It is this precise meaning that we convey by the expression, *is equal to* (or, *equals*).

Using the Correct or Preferred Term

The preciseness of mathematical language makes it possible for us to eliminate ambiguity by using correct or preferred terms and expressions.

Consider the statement, "I am bigger than you are." Does this mean that I am taller than you are? Does this mean that I am heavier than you are? The statement clearly is ambiguous in its present form. Any one of these statements would eliminate this ambiguity:

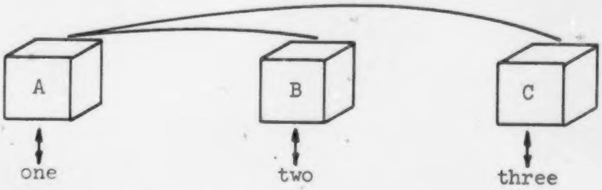
"I am taller than you are."
"I am heavier than you are."
"I am taller and heavier than you are."

In a subsequent section on "Some Important Distinctions," we shall see further illustrations of the fact that using a preferred term or expression eliminates ambiguity. Children need to be helped to choose those words that are unambiguous. One of the best ways of providing this help is by the example we set.

Multiple Meanings of Terms

Many words in our language have more than one meaning. We rely upon the context in which such a word is used to suggest the appropriate meaning in a particular situation.

Some mathematical words or expressions also have more than one meaning associated with them. An excellent example of this is seen in connection with counting the members of a set.



For instance, the word "three" may convey either or both of these ideas: it is a label or name for one particular block, block C; it also names the number property of the set whose members are blocks A, B and C. There is a place for each of these interpretations in the development of number concepts.

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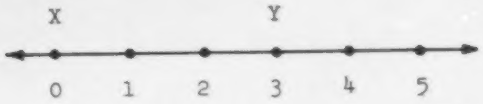
Consider also a number line representation such as this:



Here we may interpret "3" in either or both of two ways: as a designation or label for point Y, or, as indicating the measure of line segment XY (the segment having X and Y as its endpoints). Again, there is a place for each of these interpretations in the development of mathematical concepts.

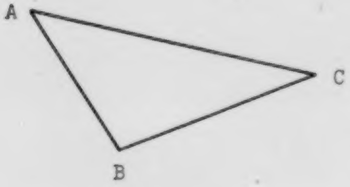
Some Important Distinctions

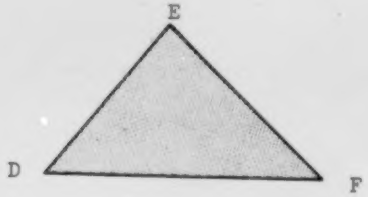
It is not uncommon in mathematical work to distinguish between terms or expressions that are often used as synonyms in "everyday language." Consider these illustrations.

1. In our work we distinguish between the expressions is equal to and is equivalent to. We previously indicated that if the statement is made, "Set A is equal to set B," this implies that "A" and "B" are names for the same set. A different meaning is implied, however, by the statement, "Set A is equivalent to set B." This latter statement implies that a one-to-one correspondence exists between the members of set A and the members of set B. For each member of A there is a member of B that can be put in correspondence with it, and for each member of B there is a member of A that can be put in correspondence with it. As we use these expressions, is equal to and is equivalent to, they are not synonymous. The fact that two sets which are equal are also equivalent, while two sets that are equivalent may or may not be equal will be discussed in Unit B.

2. In our work we distinguish between a number and a numeral. For instance, the number we refer to as "five" is a property common to a particular class of sets (e.g. the property common to all sets that are equivalent to the set of fingers, including the thumb, of your right hand). The numeral "5" is a name for that number. The characteristics of numbers and numerals are quite different.

3. In our work we distinguish between a *triangle* and a *triangular region*. For instance, the diagram at the left, below, illustrates a *triangle* (ABC). It is a union of the line segments $\overline{AB}$, $\overline{BC}$, and $\overline{CA}$.





The diagram at the right, above, illustrates a *triangular region*. It is the union of triangle DEF and its interior (represented by the shaded portion of the diagram). Clearly, *triangle DEF* is not the same thing as the *region bounded by triangle DEF*.

Distinctions such as these necessitate preciseness in the use of mathematical language and are important in clarifying certain concepts and ideas. It is not uncommon that such distinctions involve terms or expressions used synonymously in everyday language, or that they reflect distinctions not made in everyday language.

Different Language Patterns for Different Things

Some ideas which were closely associated with each other may require the use of different language patterns. Consider several ideas concerning sets of physical objects and ideas concerning numbers associated with such sets.

Set A = {a dog, a monkey, a pencil, a bottle}

Set B = {a book, an orange, a trombone}

When speaking of the sets themselves, we say that set A has *more* members than set B, or that set B has *fewer* members than set A. When speaking of the *numbers* associated with each of these sets, however, we say that 4 is *greater* than 3, or that 3 is *less* than 4. ($4 > 3$ or $3 > 4$).

If we join sets A and B to show their union, it is appropriate to state that "4 things *and* 3 things *are* 7 things." We may associate with this statement the following statement about numbers:

4 plus 3 equals 7. (4 + 3 = 7).

Here we see a difference between language patterns used with sets of physical objects and language patterns used with numbers.

Using Familiar Meanings

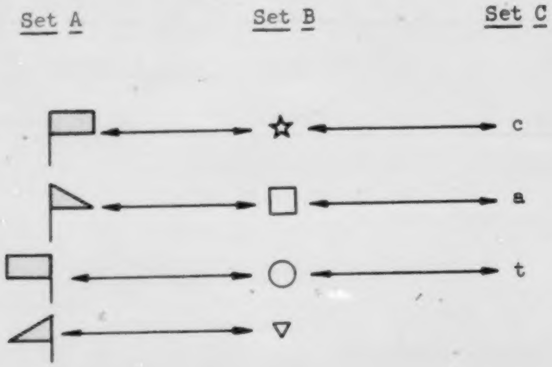
Frequently we can use familiar meanings of words to clarify their interpretation in a mathematical context. One example of this is the word member, as it occurs in the expression, "member of a set."

A child often is familiar with the fact that he is a member of his family, or that he is a member of his school class. Such instances of "member" are quite appropriate to use in developing an understanding of a member of a set. On other occasions, however, a familiar meaning of a word may not be helpful in developing an understanding of that word as used in a mathematical context. We shall see an illustration of this in the case of the word match.

Special Meanings of Familiar Terms

In their mathematical work children will encounter some familiar words that must be given a special meaning--a meaning that differs to some degree from one that applies in other contexts. Consider, for instance, the interpretation we attach to the words pair and match.

Frequently children may be asked to pair the members of two sets (in so far as it is possible to do so) to determine whether or not the two sets match (i.e. whether or not one set has exactly as many members as the other set). For example:



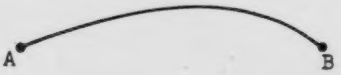
In the illustration on the previous page we can pair each member of set A with a member of set B, we can pair each member of set B with a member of set A. Thus, sets A and B are matching sets (or equivalent sets). There are as many members of one set as there are members of the other. But sets B and C do not match (i.e., they are not equivalent). Although we can pair each member of set C with a member of set B, we cannot pair each member of set B with a member of set C (and use each member of set C only once).

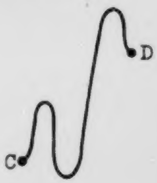
Here we see that a special meaning is given to the words *match* and *matching* as applied to sets. This meaning may differ from the way in which children interpret these words in other contexts. Similarly, *pair* and *pairing*, as applied to members of sets, convey particular meanings to us.

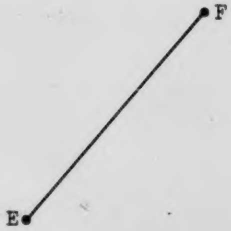
Language that Contradicts the Vernacular

We have seen that there are occasions when the mathematical interpretation of a familiar term may differ from one or another of its more general meanings. There are other occasions when the mathematical interpretation of a familiar term may even contradict its common meaning.

Consider these three representations.





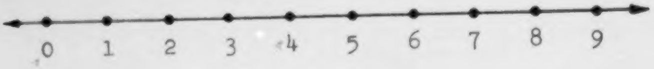


Each of these represents a *curve*. The curve at the left has A and B as its endpoints. The curve in the middle has C and D as its endpoints. The representation at the right, with E and F as endpoints, qualifies mathematically as a curve despite the fact that it is "straight." Our use of the term *curve* does *not* imply the idea of "not straight."

Here we have a good illustration of a term whose mathematical interpretation actually contradicts its common meaning under certain conditions.

Non-Mathematical Terms

In teaching mathematical ideas to young children, we must give careful attention to their understanding of words and expressions that are not mathematical but, which are relevant to their learning of mathematics. Consider, for instance, several ideas associated with a number line.



In relation to this number line, the idea of "greater than" is associated with "to the right of," and the idea of "less than" is associated with "to the left of." Specifically, 8 is greater than 5, and the point labeled 8 is to the right of the point labeled 5. Also, 3 is less than 7, and the point labeled 3 is to the left of the point labeled 7. Thus, it is crucial that children be able to distinguish between the non-mathematical terms right and left in order to interpret the mathematical relations of greater than and less than in using the "number line." Young children often have difficulty with the right-left distinction, but it must be mastered if certain mathematical ideas are to be understood. Therefore, we must give attention to those non-mathematical terms that are essential to the development of mathematical concepts.

Unnecessary Terms

Contemporary approaches to the teaching and learning of mathematics make unnecessary certain terms that in the past were a familiar part of elementary school arithmetic vocabulary. Good illustrations of this are the terms minuend, subtrahend, multiplier, and multiplicand.

Consider these number sentences: $3 + 5 = 8$ and $8 - 5 = 3$. In each instance 3 and 5 are addends, and 8 is their sum. It is not necessary to use one set of terms with the addition example and another set of terms with the subtraction example.

Also consider the number sentence, $3 \times 5 = 15$. In this instance 3 and 5 are factors whose product is 15. If we wish to distinguish between the two factors, we may refer to 3 as the "first factor" and to 5 as the "second factor." There is no need for the terms multiplier and multiplicand.

Contemporary approaches to the teaching and learning of mathematics have introduced some new terms; on the other hand, contemporary approaches to the teaching and learning of mathematics have made unnecessary certain terms that were used in the past, but frequently not understood.

CONCLUDING STATEMENT

Mathematics is a language. The teaching and learning of mathematics is therefore, the teaching and learning of a language.

The language of mathematics has its unique concepts, terminology, and symbolism. In this appendix we have attemped to highlight some of the elements that appear to be particularly crucial for culturally disadvantaged children as they learn to use this language effectively.

If we were to single out any one thing that is most important in this connection, it would be the power of the teacher's own example. Children's learning of mathematics as a language will be advanced in direct relation to the strength of the language model that you, as their teacher, set for them. It is our hope that the inservice experiences in this course will increase the strength of that model.

APPENDIX C

NUMBER CONCEPTS OF DISADVANTAGED CHILDREN

DESCRIPTION OF THE STUDY

Observation classes at the kindergarten and first grade levels were established for the 1964-65 school year in Boston, Chicago, Detroit, Miami, Oakland, and Washington, D. C. in areas described as economically and culturally disadvantaged.

This study was based on the idea that there is a differential in experiences prior to school entrance between middle-class and lower-class children. Studies from a number of sources suggest that children from more advantaged homes tend to have had experiences of greater variety in an organized family setting. By the time these children reach school-age, they appear to be better able to work in a group situation, to utilize verbal skills, and to deal with abstract concepts.

One aim of the study was to gather information on the stage of development of certain concepts relevant to the learning of mathematics in these children at the beginning of the school year and to study their growth during the year. Another aim was to discover what mathematical concepts caused the children difficulty so that a more effective program could be developed for them and to provide information to help develop materials for teachers emphasizing techniques for providing disadvantaged children with experiences necessary to make the program more effective.

The procedures adopted for gathering the information were varied and included individual and small group testing of the pupils, weekly reporting by the teachers, observations of the classrooms, and four conferences of the participating teachers.

TESTING

Individual tests were administered to the pupils in October, January, and May. A pencil and paper group test was given in June to each class. The class was split into several small groups to make it easier to administer the group test.

The table below shows what assessments were made and at which testing session for both the kindergarten and first grade pupils.

Schedule of Assessments: Individual Tests by Grade

Assessments Made	Initial		Mid-Year		Final	
	K	1st	K	1st	K	1st
Object Recognition	X	X				
Photograph Recognition	X	X				
Drawing Recognition	X	X				
Vocabulary			X	X		
Visual Memory-Objects	X	X			X	X
Visual Memory-Pictures					X	X
Color Inventory-Matching	X	X				
Color Inventory-Naming	X	X			X	X
Color Inventory-Identifying	X	X			X	X
Geometric Shapes-Matching			X	X		
Geometric Shapes-Naming			X	X	X	X
Geometric Shapes-Identifying			X	X	X	X
Pairing				X		
Equivalent Sets			X	X		
Counting Buttons	X	X	X	X	X	
Counting Sets			X	X	X	X
Rote Cardinal Counting	X	X			X	X
Rote Cardinal Counting by Tens						X
Identifying Number Symbols	X	X		X	X	X
Naming Number Symbols						X
Marking Number Symbols	X	X			X	X
Place Value-Naming						X
Place Value-Forming						X
Ordinal Number					X	X
Ordering			X	X	X	X
Classifying			X	X	X	X
Response to Verbal Directions	X	X	X	X	X	X
Attention to Tasks	X	X	X	X	X	X

In the space available it is not possible to include all the results from each session. An analysis of the results is contained in a separate report of the project. A selection has been made of those results that highlight certain points in the learning of mathematical concepts of these disadvantaged children. The specific areas included are object recognition, color, geometric shapes, rational counting and the recognition and writing of number symbols. These are typical of the level and range of abilities found in all the classes tested and point out clearly some of the problems that the teachers of disadvantaged children encounter.

OBJECT RECOGNITION

In this section, the pupil's ability to recognize objects and to recognize pictorial representations of such objects that are used in the curriculum materials, was measured. The child was shown an object, a drawing of an object, and a photograph of an object and asked, "What is this?" The number of items and the approximate mean score for each of these assessments were as follows:

Assessment	Number of Items	Approximate Mean
Object recognition	23	20
Photograph recognition	10	9
Drawing recognition	7	6

Little difference was found between the classes in the disadvantaged areas and classes in middle-class areas.

Objects causing difficulty were different fruit, coins, and string. From 10-20% of the pupils were unable to name orange; banana 10% ; penny 10% ; nickel 20-30% ; dime 20-25% ; and string 20-40%. Although many could not name string, they could indicate what function it served.

The results indicate that most pupils are able to name and identify the objects suggested in the text materials for use in the classroom.

Do not infer from these results, however, that the verbal skills and experiences of the two groups are the same. It has already been pointed out that children from disadvantaged groups will lack many of the experiences which facilitate school learning.

COLOR AND GEOMETRY

Children from all backgrounds are able to match the basic colors, but their ability to name these colors and to select a color when given its name are very variable at the beginning of kindergarten. Children from middle-class areas are fairly proficient on these two tasks when entering first grade but children from disadvantaged areas are not. Typical results for classes in the same city were:

	Number of Items	KINDERGARTEN		FIRST GRADE	
		E	C	E	C
Matching	6	5.4	5.7	5.6	6.0
Naming	7	4.9	5.8	6.1	6.9
Identifying	6	4.1	4.9	5.3	6.0

[E denotes experimental classes (disadvantaged area) and
C denotes comparison classes (middle class).]

The results from this and other inventories point out the need for teachers to be aware of the three-fold nature of many of the tasks that children must learn at these grade levels. The child has to be able to match two objects, e.g., the name "three" to the numeral "3" and, when given a number name, to be able to select the correct number of objects. The assessments showed generally that matching was the easiest, then identifying, and naming the most difficult. However, by continually providing the children with experiences in the three phases, considerable improvement can be seen. The results in geometry for the same first grade classes from the mid-year to final inventories show this.

GEOMETRIC SHAPES - CORRECT RESPONSES

First Grade

	Mid-Year		Final	
	E	C	E	C
Matching				
Circle	96%	100%	- -	- -
Square	96%	96%	- -	- -
Triangle	96%	100%	- -	- -
Rectangle	96%	100%	- -	- -
Mean	3.8	4.0	- -	- -
Naming				
Square	39%	93%	81%	93%
Triangle	57%	93%	96%	79%
Rectangle	0%	85%	50%	64%
Circle	50%	81%	73%	89%
Mean	2.2	2.5	3.3	3.1

	Mid-Year		Final	
	E	C	E	C
Identifying				
Triangle	89%	93%	100%	89%
Rectangle	14%	85%	77%	89%
Circle	96%	96%	92%	100%
Square	64%	93%	85%	96%
Mean	3.1	3.2	3.8	3.8

This table shows the gains that pupils in the experimental first grade classes were able to make in naming and identifying the geometric shapes. In both of these tasks, the means for the experimental group were lower than the means for the comparison classes in the midyear assessment. For the final inventories, the mean for the experimental classes in these tasks were at least as great as those for the comparison group. These gains may be seen also in the table below which shows the frequency distribution of correct responses for the same classes in naming and identifying the geometric shape. It should be noted that although the pupils improved, there remained more variability within the experimental class.

NAMING AND IDENTIFYING GEOMETRIC SHAPES

Number of Tasks Successfully Completed	Naming				Identifying			
	Mid-Year		Final		Mid-Year		Final	
	E	C	E	C	E	C	E	C
0	6	0	0	0	1	2	0	0
1	6	0	3	0	1	0	0	0
2	13	2	3	8	8	2	4	2
3	3	9	11	5	15	1	4	2
4	0	16	9	15	3	23	18	24

COUNTING MEMBERS OF A GIVEN SET

Several different tasks were used to measure the pupil's ability to count given sets of objects and to select given numbers of objects. Other number tasks such as rote counting were also included in the assessments. There is little evidence to suggest however that a child who can count by rote will necessarily be able to count the members of a set.

One task pupils were asked to do was to count out 3 buttons, then 5 buttons, 4 buttons, 6 buttons, 8 buttons, 7 buttons, and 9 buttons. Initially, the average kindergarten pupil could complete successfully only two or three of these tasks. At the end of the school year, this pupil could manage about six tasks correctly. Similar improvement was found with first grade pupils from the initial inventory (five to six of these tasks correct) to mid-year when the average was almost seven correct. The greatest difference found between the experimental and control classes was the greater variability among students in the experimental classes.

COUNTING BUTTONS

Number of tasks Successfully completed	Kindergarten				First Grade			
	Initial		Final		Initial		Mid-Year	
	E	C	E	C	E	C	E	C
0	2	3	0	0	2	0	2	0
1	12	3	0	3	0	0	1	0
2	4	1	0	0	2	0	0	1
3	5	2	2	0	1	0	0	0
4	2	6	1	0	2	2	0	0
5	3	2	1	0	0	5	1	1
6	1	1	2	3	6	0	4	1
7	1	6	20	17	16	21	20	24

The variability in level of performance between pupils from disadvantaged areas was not confined to number tasks but was also evident in other tasks that they were required to do. For example, the table below shows the frequency distribution of correct responses on color naming for two classes of first grade children at the end of the year. The number of tasks performed successfully by the first graders in the comparison group was concentrated near 7, whereas there was a greater spread in the experimental group. This is reflected in the standard deviation of 0.7 for the experimental group and of 0.1 for the comparison group.

NAMING COLORS

First Grade

Number Correct	Experimental	Comparison
0	1	0
1	0	0
2	0	0
3	0	0
4	3	1
5	4	1
6	10	0
7	11	26

RECOGNITION AND WRITING NUMBER SYMBOLS

First grade teachers often assume that most of their pupils can recognize, name, and also write many of the numerals when they start first grade. However, children in disadvantaged areas will generally not be able to do any of these tasks well, as the following two tables show.

RECOGNITION AND WRITING NUMBER SYMBOLS - PERCENT CORRECT

Numeral	Recognition		Write	
	E	C	E	C
0	79	100	---	---
3	83	100	48	79
4	76	100	24	100
5	76	100	21	100
6	65	100	31	89
7	69	100	24	68
8	69	100	45	89
9	59	100	14	79

In identifying these number symbols, the mean percent for the experimental group was 75 and for the comparison group, 100. In writing the number symbols, the mean percent was 29 for the experimental as against 86 for the comparison. Here again, there was greater variability in the experimental group as can be noted in the table for the frequency distribution. In recognition (identifying) of number symbols, the standard deviations are 2.9 for the experimental group and 0 for the comparison group. In writing number symbols, the standard deviations are 2.0 for the experimental and 1.3 for the comparison.

FREQUENCY DISTRIBUTION

Number Correct	Recognition		Write	
	E	C	E	C
0	2	0	10	--
1	3	0	5	--
2	1	0	2	--
3	1	0	5	2
4	0	0	2	2
5	0	0	2	4
6	3	0	3	5
7	3	0	0	15
8	16	28	--	--

Although the recognition and writing of numerals are NOT fundamental to the understanding of mathematical concepts, these are skills essential for communicating mathematical ideas and concepts, and the child who lacks them will be handicapped for future learning.

All the pupils will need additional practice in writing numerals, but this practice should not be given until they have an understanding of the elementary concepts of number. It cannot be assumed that if children can recognize the numerals they can write them. This can be seen in the difference between the means mentioned above. The children will need a great deal of careful practice which can be given independently of the mathematics lesson.

In the early stages an adhesive number line attached to pupils' desks is a very useful aid as they can use it whenever they need it. Even with number lines on their desks there may be some pupils who will not be able to form the numerals correctly, e.g., "7" will be made as "Γ" and "9" as "∂" and the pupils will need additional assistance.

GENERAL DISCUSSION

One fact that clearly emerges from the assessments made this year is that it was difficult to predict from the initial test what development and progress a pupil will make during the year. Some pupils whose scores on the initial assessment were low, progressed very rapidly; others did not. All pupils did make progress but at variable rates. This is one of the major problems that confronts the teacher of disadvantaged children. The range of abilities in these classes is much wider than found in classes of middle-class children, as has already been noted.

A technique which the teachers found useful this year was to observe carefully three selected pupils each day. These systematic observations gave the teacher valuable information about the strengths, weaknesses and difficulties of each pupil and enabled her to chart changes in his level of performance.

In attempting to meet this variability, some teachers grouped pupils for instruction as is often done in reading, while others kept the class as a unit but provided work sheets which took into account the different performance levels of the pupils. If grouping is used, it is essential to move an individual from group to group as he changes in his level of performance. The success of any method depends upon the pupils being able to use blocks or other manipulative material to do their work sheets while the teacher works with a small group or an individual.

As yet there are no adequate materials nor work sheets that will keep all of the pupils fully and profitably engaged at all times. This is a problem and challenge that each teacher must face. However, a great deal can be done and a great deal more can be learned if each teacher tries a variety of approaches and reports her successes and failures.

ANSWERS TO EXERCISES

CHAPTER 1

1. a. {Wednesday}
   b. {pitcher, catcher, first base, second base, shortstop, third base, left field, center field, right field}
   c. {March, April, May, June, July}
   d. { } or ∅
   e. Answers will vary. Example {5, 6, 7}
   f. {Tokyo, London}
   g. {red, orange, yellow, green, blue, indigo, violet}

2. a. {49th and 50th states of U. S. A.}
   b. {things little boys are made of}

3. They are all members of the cat family.

4. a. Not equal. 17 and 71 are names for different numbers.
   b. Equal. The sets are the same. Order makes no difference, even though it would be more natural to write {b, u, n, d, l, e}.
   c. Equal. The same elements are listed; order is irrelevant.
   d. Not equal. {zero} has a single member, as opposed to { } which has none. {peacocks native to the North Pole} is the empty set.
   e. Not equal. The members are different.
   f. Not equal. These two sets each have single members but they are not the same member. "Are" and "era" are not names for the same thing.
   g. Equal. Remember. Elements are not listed more than once in a set.

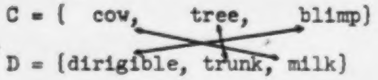
5. a. False. 3 is a member of {1, 2, 3}. The braces must be used to indicate set. A correct statement would be: "{3} is a subset of {1, 2, 3}".
   b. True. {ego} is a subset because all of its members are also members of {ego, je, I}.

c. True. Any set equals itself, and every set is a subset of itself.

d. False. There are birds that are not hens, a rooster to name one. It would be correct to say {all hens in the world} is a subset of {all birds in the world}.

6. a. {rose, tulip, dandelion} or any subset of this.
   b. {bee, beetle} or any of its subsets.
   c. { }. Some may consider {beetle} but the spelling is different.

CHAPTER 2

1. a. A has fewer members than B
   b. 
   C = { cow, tree, blimp}
   D = {dirigible, trunk, milk}

   Answers may vary.

2. Z, X, Y in increasing order.
   Y, X, Z in decreasing order.

3. Mary is taller than Andrea.

4. Q has more members than P.

5. a. and c. only.
   a. To show the 1-1 correspondence, the natural pairing would associate each person and his brain. The question of what is meant by "functioning" brains may enter into consideration. Depending on the answer to this, it may be that there is no 1-1 correspondence.
   b. This is not necessarily one-to-one. Some people have social security numbers and don't file income tax returns. There are many joint income tax returns filed. Corporations file returns but have no social security number. It is conceivable that these two sets might be equivalent, although the natural pairing would not show it.
   c. E is the empty set and so is F. Therefore, E and F are equal. Every set matches itself.
   d. G = {seeing, hearing, tasting, smelling, touching}.
   H = {5}. These sets are obviously not in 1-1 correspondence.

6. A is equivalent to B.
   The members of A and B are in 1-1 correspondence. The members of A can be paired with those of B with none left over.

CHAPTER 3

1. a. 4 b. 1 c. 1
   d. 2; the set is {d, e}.
   e. 1; {the vowels in "bureau"} = {u, e, a}. However, this set is {the number of vowels in "bureau"} which is the single member set {3} whose number property is 1.
   f. 0; there are no counting numbers less than one, so this is the empty set.

2. N(A) = 4 N(B) = 3 N(C) = 0 N(D) = 3
   The relationships are: $3 = 3$ $0 = 0$ $4 = 4$
   $4 > 3$ $3 < 4$
   $4 > 0$ $0 < 4$
   $3 > 0$ $0 < 3$
   Some statements can be combined to form two others:
   $0 < 3 < 4$ and $4 > 3 > 0$.

3. Take a set of wide and narrow objects of the same kind. Put the wide ones in a set by themselves and reject the narrow ones. Repeat until the idea gets across, refining distinctions to indicate the relativity of "wide" and "narrow" to some standard. This is an instance of specifying sets according to the property you wish to convey.

4. Answers may vary. The elements may be ordered alphabetically; they may be ordered according to the evolutionary development of man: {amoeba, fish, lizard, ape, man}.

5. {1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11}. The number is 11. It is now written as a standard subset of the counting numbers whose number is determined by the last member of the set.

6. a. Finite; 10
   b. Infinite
   c. Infinite
   d. Finite; 15. (Recall that "natural numbers" is another name for "counting numbers".)

7. a. Ordinal; one chapter, not the quantity of three chapters, is referred to, namely, the third chapter.
   b. Cardinal; 50 states are mentioned, not the one which is fiftieth.
   c. Ordinal; 1066 suggests one year, not one thousand sixty-six years. It happens to be the one-thousand sixty-sixth year in the set of years A. D.

CHAPTER 4

1. a. $A \cup B = \{1, 2, 3, 4, 5\}$
   $A \cap B = \{1, 3, 5\}$
   b. If B is a subset of A, then $A \cup B = A$
   c. If B is a subset of A, then $A \cap B = B$

2. The empty set is a subset of any set so if A is a set, $A \cup \{\ \} = A$ and $A \cap \{\ \} = \{\ \}$

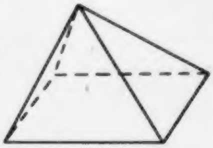
3. The cake-mixing operation is not commutative.

4. a. Associative.
   b. Not associative; although the result may be that all items would be consumed, it is likely that mixing mustard with coffee may result in abandoning the meal.
   c. Associative.
   d. Not associative; putting fire with water first, the final mixture will not ignite.

*5. The intersection consists of common elements of both sets; the union contains all members of both sets.

CHAPTER 5

1. Five vertices.



2. $\overline{AB}$ is the segment with A and B as endpoints. $\overrightarrow{AB}$ is the ray with A as endpoint and B a point in the ray, $\overleftrightarrow{AB}$ is the line containing the points A and B.

3. Infinitely many different lines may contain a certain point; only one line contains a certain pair of points.

4. a, b, and c are all true.

5. Any segment contains the endpoints. If $\overline{PQ}$ is divided at say, R, the division point either belongs to the segment containing P or to the segment containing Q. If R belongs to both, then $\overline{PR} \cup \overline{RQ} = \overline{PQ}$ but $\overline{PR}$ and $\overline{RQ}$ are not disjoint. If R belongs to $\overline{PR}$, then the set of points in $\overline{RQ}$ without R is not a segment. Crucial to the argument is that for points in a line, there is no very next point; so if R belongs to $\overline{PR}$, there is no next point S to specify $\overline{SQ}$ so that $\overline{PR} \cup \overline{SQ} = \overline{PQ}$.

6. a. In this closed curve, each point is between the other two; there is no one point that is between the others.
   b. No, as noted in a. above.

7. a. The initial point and the terminal point.
   b. A closed curve, as for example, formed by the shore line and a ring of blockading armada.
   c. A closed surface, as for example, determined by effective range of antiaircraft defense network.
   d. A closed surface with the sea surface as one of the boundaries.

8. The point associated with the larger number is to the right.

9. a. Grades 1, 2, 3. Closed curves is a section title in Book 1 (Chapter V-2), Book 2 (Chapter III-4), and Book 3 (Chapter I-3).
   b. Curves as a basic concept for closed curves is a topic in Book K (Chapter II). Topics using the basic concepts of closed curves are, for example, polygons, Book 1 (Chapter V-3); triangles, Book 2 (Chapter III-5); Book 3 (Chapter I-3,I-5,III-4).

CHAPTER 6

1. a. >
   b. <
   c. <
   d. >
   e. =
   f. >

2. a. 8
   b. 20
   e. 38
   d. 39
   e. 65
   f. 156

CHAPTER 7

1. a. 7
   b. 17
   c. 23
   d. 60
   e. 60
   f. 23

2. c and f; d and e; because addition of whole numbers is commutative

3. a. Identity Property
   b. Associative Property
   c. Commutative Property

4. a. 7
   b. 23
   c. 6
   d. 401

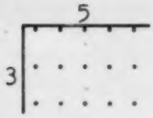
5. a. Commutative
   b. Identity
   c. Associative
   d. Commutative
   e. Identity
   f. Commutative

CHAPTER 8

1. $1000 \times 3 = \overbrace{3 + 3 + 3 + 3 + \dots + 3}^{1000\ \text{addends}} = 3000$. This expresses 1000×3. By the commutative property of multiplication, $1000 \times 3 = 3 \times 1000$, and $3 \times 1000 = 1000 + 1000 + 1000 = 3000$.

2. a. $4 \times 5 = 20$;
   c. $2 \times 4 = 8$;
   b. $3 \times 2 = 6$;
   c. $3 \times 3 = 9$

3.



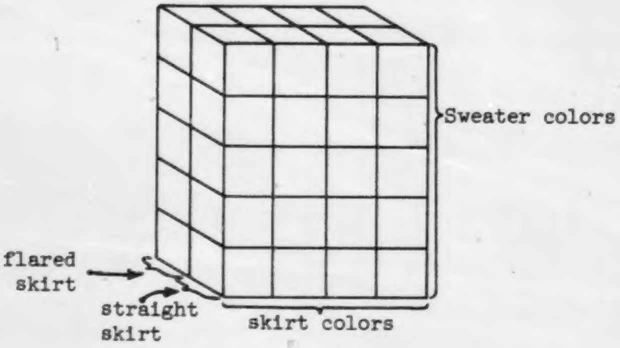
4.

	red	orange	yellow	green	blue
red	red red	orange red	yellow red	green red	blue red
yellow	red yellow	orange yellow	yellow yellow	green yellow	blue yellow
blue	red blue	orange blue	yellow blue	green blue	blue blue

15 possible results.

If the car must be two-toned, there are only 12 choices.

5.


Sweater colors
flared skirt
straight skirt
skirt colors

$2 \times 4 \times 5 = 40$.

40 different ensembles.

6. a. $n = 32$; $p = 12$; $q = 20$

b. yes ; c. yes

7. The star pattern does not give 5 disjoint sets with 4 members in each set.

8. $20 \times (28 + 11 + 11) = (20 \times 28) + (20 \times 11) + (20 \times 11)$

$= 560 + 220 + 220$

$= 560 + 440$

$= 1000$

or $20 \times (28 + 11 + 11) = 20 \times (39 + 11)$

$= 20 \times (50)$

$= 1000$

9. a. $(5 \times 2) \times (4 \times 3) \times 1 = 10 \times 12 = 120$

b. $(125 \times 8) \times (7 \times 3) = 1000 \times 21 = 21{,}000$

c. $(250 \times 4) \times (14 \times 2) = 1000 \times 28 = 28{,}000$

10. Commutative property under multiplication.

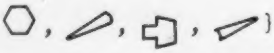
11. a. $3 \times (4 + \underline{3}) = (3 \times 4) + (3 \times 3)$

b. $2 \times (\underline{4} + 5) = (2 \times 4) + (\underline{2} \times 5)$

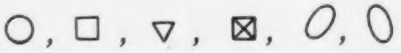
c. $13 \times (16 + 4) = (13 \times \underline{16}) + (13 \times \underline{4})$

d. $(2 \times 7) + (3 \times \underline{7}) = (\underline{2} \times \underline{3}) \times 7$

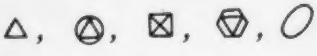
CHAPTER 9

1. C = {  }

Joining C to B yields $B \cup C = A$.

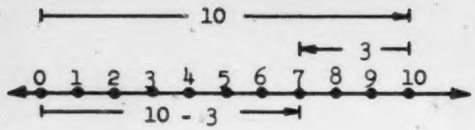
2. A ~ B = {  }

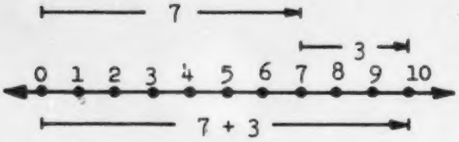
3. 6

4. B = {  }

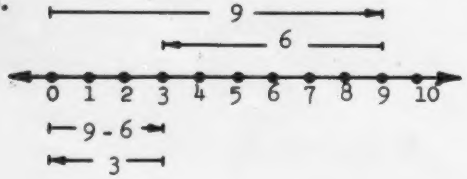
N(B) = 5

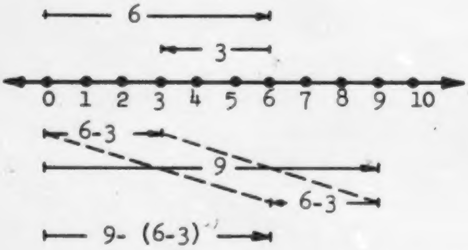
5.


10
3
0 1 2 3 4 5 6 7 8 9 10
10 - 3


7
3
0 1 2 3 4 5 6 7 8 9 10
7 + 3

6.


9
6
0 1 2 3 4 5 6 7 8 9 10
9 - 6
3


6
3
0 1 2 3 4 5 6 7 8 9 10
6-3
9
6-3
9- (6-3)

7. Subtracting 7 from the sum.

Adding 8 to the difference.

8. Let A = { ○ , △ , □ } and B = {a , b , c} .
Then A⋃B = { ○ , △ , □ , a , b , c}
and (A⋃B) - B = { ○ , △ , □ } = A .

If A and B are not disjoint, the sets (A⋃B) - B and A are not equal. See example.

A = {a , b , c , d , e} ; B = {a , d , g , j} .
A⋃B = {a , b , c , d , e , g , j} .
(A⋃B) - B = {b , c , e} , which is a new set.

CHAPTER 10

1. a. $n = 20 \div 5$; $n = 4$
   b. $p = 28 \div 4$; $p = 7$
   c. $n = 6 \div 1$; $n = 6$
   d. $n = 72 \div 9$; $n = 8$
   e. $n = 64 \div 8$; $n = 8$
   f. No division sentence can be written. Division by 0 is undefined. $q \times 0 = 0$ is true for any number q.

2. a. Rectangular array with 7 rows and 6 columns.
   b. Disjoint subsets, six with seven members each.

   Either interpretation is equally valid. There may be slight preference in thinking of disjoint subsets in b, since subsets of seven members each are specified in the packaging.

3. The number 59 is a prime number, so no rectangular arrays can be formed other than one with a single row or a single column. Sixty members allows many rectangular formations since its factors are 1, 2, 3, 4, 5, 6, 10 12, 15, 20, 30, 60.

4. No. $15 \div 5 \neq 5 \div 15$. In fact, $5 \div 15$ has no meaning in the set of whole numbers.

5. a. 2 × 6 or 3 × 4
   b. 2 × 18 ; 3 × 12 ; 4 × 9 ; or 6 × 6
   c. Prime
   d. Prime
   e. 2 × 4
   f. Prime
   g. 5 × 7
   h. Prime
   i. 3 × 13
   j. 2 × 21 ; 3 × 14 ; or 6 × 7
   k. 2 × 3
   l. Prime
   m. 2 × 41
   n. 5 × 19

6. a. $2 \times 2 \times 2 \times 2$
   b. 3×7
   c. $3 \times 3 \times 7$
   d. $2 \times 3 \times 3 \times 5$
   e. $2 \times 2 \times 2 \times 2 \times 3 \times 3$
   f. $2 \times 2 \times 3 \times 11$

7. a. 1
   b. 1
   c. 2
   d. 3
   e. 1
   f. 4
   g. 3
   h. 8

CHAPTER 11

1. If $A = C$, then $\overrightarrow{AB} \cup \overrightarrow{CD}$ is a point; if $A \neq C$ then the union is a segment.

2. $\overline{AB}$ is the segment with A and B as endpoints.
   $\overrightarrow{AB}$ is the ray with A as endpoint and B a point in the ray;
   $\overleftrightarrow{AB}$ is the line containing the points A and B.

3. $\angle PQR$; $\angle PQS$; $\angle PQT$; $\angle RQS$; $\angle RQT$; $\angle SQT$.

4. a, d, c

5. a, c, d

CHAPTER 12

1. a.

$$\begin{array}{lcl} 2 \text{ hundreds} + 4 \text{ tens} + 6 \text{ ones} & \text{or} & 200 + 40 + 6 \\ \underline{1 \text{ hundred} + 3 \text{ tens} + 9 \text{ ones}} & & \underline{100 + 30 + 9} \\ 3 \text{ hundreds} + 7 \text{ tens} + 15 \text{ ones} & & 300 + 70 + 15 \\ 3 \text{ hundreds} + 8 \text{ tens} + 5 \text{ ones} = 385 & & 300 + 80 + 5 = 385 \end{array}$$

or

$$\begin{array}{r} 246 \\ \underline{139} \\ 15 \\ 70 \\ \underline{300} \\ 385 \end{array}$$

b.

$$\begin{array}{rcrcrl}
7 \text{ hundreds} & + & 7 \text{ tens} & + & 7 \text{ ones} & \\
9 \text{ hundreds} & + & 6 \text{ tens} & + & 4 \text{ ones} & \\
\hline
16 \text{ hundreds} & + & 13 \text{ tens} & + & 11 \text{ ones} & \\
17 \text{ hundreds} & + & 4 \text{ tens} & + & 1 \text{ ones} & = 1741
\end{array}
\qquad \text{or} \qquad
\begin{array}{rcrcrl}
700 & + & 70 & + & 7 & \\
900 & + & 60 & + & 4 & \\
\hline
1600 & + & 130 & + & 11 & \\
1700 & + & 40 & + & 1 & = 1741
\end{array}$$

or

$$\begin{array}{r}
777 \\
\underline{964} \\
11 \\
130 \\
\underline{1600} \\
1741
\end{array}$$

c.

$$\begin{array}{rcrcrl}
4 \text{ hundreds} & + & 8 \text{ tens} & + & 6 \text{ ones} & \\
7 \text{ hundreds} & + & 6 \text{ tens} & + & 6 \text{ ones} & \\
\hline
11 \text{ hundreds} & + & 14 \text{ tens} & + & 12 \text{ ones} & \\
12 \text{ hundreds} & + & 5 \text{ tens} & + & 2 \text{ ones} & = 1252
\end{array}
\qquad \text{or} \qquad
\begin{array}{rcrcrl}
400 & + & 80 & + & 6 & \\
700 & + & 60 & + & 6 & \\
\hline
1100 & + & 140 & + & 12 & = 1252
\end{array}$$

or

$$\begin{array}{r}
486 \\
\underline{766} \\
12 \\
140 \\
\underline{1100} \\
1252
\end{array}$$

d.

$$\begin{array}{rcrcrl}
7 \text{ hundreds} & + & 7 \text{ tens} & + & 4 \text{ ones} & \\
9 \text{ hundreds} & + & 2 \text{ tens} & + & 6 \text{ ones} & \\
\hline
16 \text{ hundreds} & + & 9 \text{ tens} & + & 10 \text{ ones} & \\
17 \text{ hundreds} & + & 0 \text{ tens} & + & 0 \text{ ones} & = 1700
\end{array}
\qquad \text{or} \qquad
\begin{array}{rcrcrl}
700 & + & 70 & + & 4 & \\
900 & + & 20 & + & 6 & \\
\hline
1600 & + & 90 & + & 10 & \\
1700 & + & 0 & + & 0 & = 1700
\end{array}$$

or

$$\begin{array}{r}
774 \\
\underline{926} \\
10 \\
90 \\
\underline{1600} \\
1700
\end{array}$$

2. a.

$$\begin{array}{l}
7 \text{ hundreds} + 6 \text{ tens} + 4 \text{ ones} = 6 \text{ hundreds} + 15 \text{ tens} + 14 \text{ ones} \\
\underline{1 \text{ hundred} + 9 \text{ tens} + 9 \text{ ones}} = \underline{1 \text{ hundred} + 9 \text{ tens} + 9 \text{ ones}} \\
\phantom{1 \text{ hundred} + 9 \text{ tens} + 9 \text{ ones} = {}} 5 \text{ hundreds} + 6 \text{ tens} + 5 \text{ ones} = 565
\end{array}$$

or

$$\begin{array}{l}
700 + 60 + 4 = 600 + 150 + 14 \\
\underline{100 + 90 + 9} = \underline{100 + 90 + 9} \\
\phantom{100 + 90 + 9 = {}} 500 + 60 + 5 = 565
\end{array}$$

b.
$$\begin{array}{lcl}
4 \text{ hundreds} + 0 \text{ tens} + 2 \text{ ones} & = & 3 \text{ hundreds} + 9 \text{ tens} + 12 \text{ ones} \\
\underline{1 \text{ hundred} + 3 \text{ tens} + 8 \text{ ones}} & = & \underline{1 \text{ hundred} + 3 \text{ tens} + 8 \text{ ones}} \\
 & & 2 \text{ hundreds} + 6 \text{ tens} + 4 \text{ ones} = 264
\end{array}$$

or
$$\begin{array}{lcl}
400 + 0 + 2 & = & 300 + 90 + 12 \\
\underline{100 + 30 + 8} & = & \underline{100 + 30 + 8} \\
 & & 200 + 60 + 4 = 264
\end{array}$$

c.
$$\begin{array}{lcl}
7 \text{ hundreds} + 1 \text{ ten} + 0 \text{ ones} & = & 6 \text{ hundreds} + 10 \text{ tens} + 10 \text{ ones} \\
\underline{2 \text{ hundreds} + 8 \text{ tens} + 7 \text{ ones}} & = & \underline{2 \text{ hundreds} + 8 \text{ tens} + 7 \text{ ones}} \\
 & & 4 \text{ hundreds} + 2 \text{ tens} + 3 \text{ ones} = 423
\end{array}$$

or
$$\begin{array}{lcl}
700 + 10 + 0 & = & 600 + 100 + 10 \\
\underline{200 + 80 + 7} & = & \underline{200 + 80 + 7} \\
 & & 400 + 20 + 3 = 423
\end{array}$$

d.
$$\begin{array}{lcl}
8 \text{ hundreds} + 0 \text{ tens} + 0 \text{ ones} & = & 7 \text{ hundreds} + 9 \text{ tens} + 10 \text{ ones} \\
\underline{3 \text{ hundreds} + 9 \text{ tens} + 6 \text{ ones}} & = & \underline{3 \text{ hundreds} + 9 \text{ tens} + 6 \text{ ones}} \\
 & & 4 \text{ hundreds} + 0 \text{ tens} + 4 \text{ ones} = 404
\end{array}$$

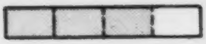
or
$$\begin{array}{lcl}
800 + 0 + 0 & = & 700 + 90 + 10 \\
\underline{300 + 90 + 6} & = & \underline{300 + 90 + 6} \\
 & & 400 + 0 + 4 = 404
\end{array}$$

3.
$$\begin{aligned}
774 + 926 &= (700 + 70 + 4) + (900 + 20 + 6) \\
&= (700 + 900) + (70 + 20) + (4 + 6) \\
&= 1600 + 90 + 10 \\
&= 1600 + 100 \\
&= 1700
\end{aligned}$$

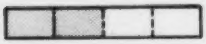
4.
$$\begin{aligned}
800 - 396 &= 800 - (300 + 90 + 6) \\
&= (700 + 90 + 10) - (300 + 90 + 6) \\
&= (700 - 300) + (90 - 90) + (10 - 6) \\
&= 400 + 4 \\
&= 404
\end{aligned}$$

CHAPTER 13

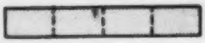
1. a.



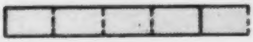
b.



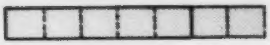
c.



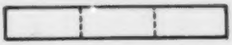
d.



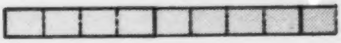
e.



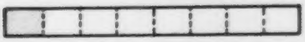
f.



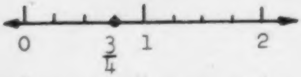
g.



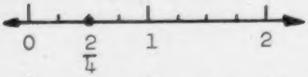
h.



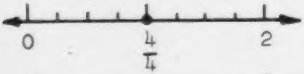
2. a.



b.



c.



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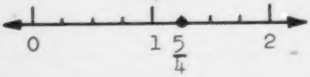
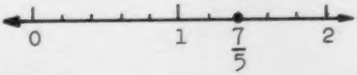


e.



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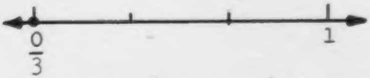

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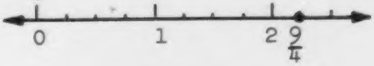
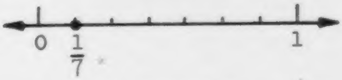
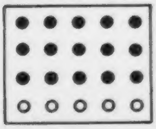


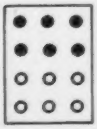
h.



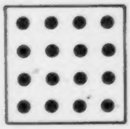
3. a.



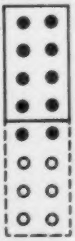
b.



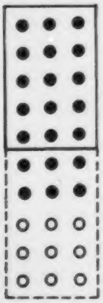
c.



d.

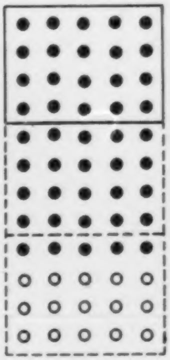


e.

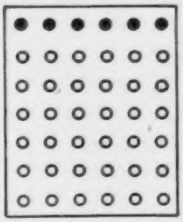


f. (empty)

g.



h.



4. a. $\frac{6}{4}$

b. $\frac{2}{8}$

c. $\frac{3}{8}$

d. $\frac{4}{6}$

e. $\frac{6}{10}$

f. $\frac{3}{2}$

g. not an appropriate model

h. not an appropriate model

i. not an appropriate model

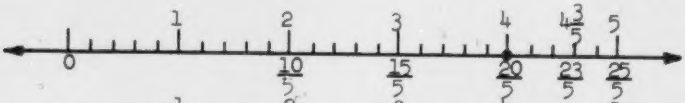
j. $\frac{6}{6}$

5. a. A , $\frac{1}{2}$ or $\frac{2}{4}$; B , $\frac{1}{4}$; C , $\frac{3}{4}$; D , $\frac{1}{3}$; E , $\frac{2}{3}$

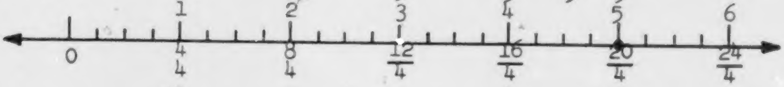
b. less than, since B lies to the left of D while 1 lies to the right of 0 .

c. $\frac{1}{2}$ or $\frac{2}{4}$

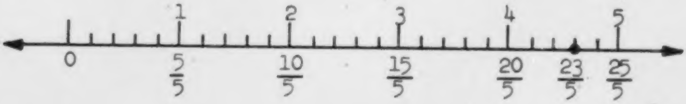
6. a.


0
1
2
3
4
$4\frac{3}{5}$
5
$\frac{10}{5}$
$\frac{15}{5}$
$\frac{20}{5}$
$\frac{23}{5}$
$\frac{25}{5}$

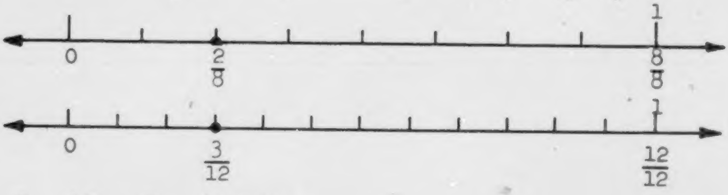
b.


0
1
2
3
4
5
6
$\frac{4}{4}$
$\frac{8}{4}$
$\frac{12}{4}$
$\frac{16}{4}$
$\frac{20}{4}$
$\frac{24}{4}$

c.


0
1
2
3
4
5
$\frac{5}{5}$
$\frac{10}{5}$
$\frac{15}{5}$
$\frac{20}{5}$
$\frac{23}{5}$
$\frac{25}{5}$

7.


0
$\frac{2}{8}$
1
$\frac{8}{8}$
0
$\frac{3}{12}$
1
$\frac{12}{12}$

8. $\frac{1}{4}$, $\frac{7}{12}$, $\frac{12}{13}$, $\frac{7}{412}$, $\frac{412}{7}$, $\frac{2}{3}$

9. a. $\frac{1}{25} < \frac{1}{24}$

b. $\frac{11}{24} < \frac{12}{26}$

c. $\frac{7}{8} > \frac{5}{6}$

d. $\frac{17}{32} > \frac{1}{2}$

e. $\frac{13}{26} = \frac{9}{18}$

10. a. $1\frac{3}{4}$

b. $1\frac{7}{8}$

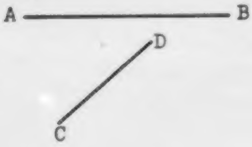
c. $2\frac{3}{9} = 2\frac{1}{3}$

d. $2\frac{4}{15}$

e. $4\frac{8}{12} = 4\frac{2}{3}$

CHAPTER 14

1. Since segments have two endpoints, it is quite possible for them not to intersect and yet not lie in parallel lines. $\overline{AB}$ and $\overline{CD}$ illustrate two segments which satisfy the conditions of lying in the same plane and not intersecting; however, they are not parallel.

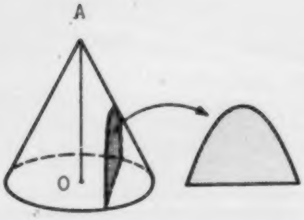


2. The line; a point; { } .

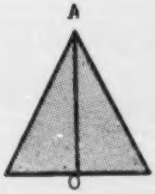
3. Model construction.

4. a. 6 ; b. 8 ; c. 2 × n

5.



or



if the plane contains the line of center

6. a. ; b.
   c. does not have to be. When the quadrilateral is not convex, the pyramid is not.

7. $\overline{XZ}$, $\overline{ZY}$, $\overline{XY}$

8. $\overrightarrow{AB}$ contains the point A ; A is in the angle, not in its interior.

CHAPTER 15

1. These answers are illustrative; others are possible.

a. $\begin{array}{l} (30 + 4) \\ \times \qquad 7 \\ \hline 210 + 28 \\ 230 + 8 = 238 \end{array}$ or $\begin{array}{r} 34 \\ \times 7 \\ \hline 28 \\ 210 \\ \hline 238 \end{array}$ b. $\begin{array}{r} 48 \\ \times 6 \\ \hline 48 \\ 240 \\ \hline 288 \end{array}$ or $\begin{array}{r} 48 \\ \times 6 \\ \hline 288 \end{array}$

c. $\begin{array}{l} (20 + 8) \\ \times \qquad 9 \\ \hline 180 + 72 \\ 250 + 2 = 252 \end{array}$ or $\begin{array}{r} 20 \\ \times 9 \\ \hline 180 \end{array}$ $\begin{array}{r} 8 \\ \times 9 \\ \hline 72 \end{array}$ $\begin{array}{r} 180 \\ + 72 \\ \hline 252 \end{array}$

d. $\begin{array}{r} 54 \\ \times 8 \\ \hline 32 \\ 40 \\ \hline 432 \end{array}$ or $\begin{array}{r} 54 \\ \times 8 \\ \hline 432 \end{array}$

2. a. $38 = (6 \times 6) + 2$; also, $38 \div 6 = 6\frac{2}{6} = 6\frac{1}{3}$

b. $99 = (24 \times 4) + 3$; also, $99 \div 4 = 24\frac{3}{4}$

c. $125 = (15 \times 8) + 5$; also, $125 \div 8 = 15\frac{5}{8}$

d. $84 = (28 \times 3)$; also, $84 \div 3 = 28$

3. a. $\begin{array}{r} 48 \\ 7\overline{)342} \\ 28 \\ \hline 62 \\ 56 \\ \hline 6 \end{array}$ b. $\begin{array}{r} 48 \\ 7\overline{)342} \\ 28 \\ \hline 62 \\ 56 \\ \hline 6 \end{array} \begin{array}{c} \\ \\ \rightarrow \\ \\ \rightarrow \\ \rightarrow \end{array} \begin{array}{r} \\ 7\overline{)342} \\ -\ 280 \\ \hline 62 \\ -\ 56 \\ \hline 6 \end{array} \quad \begin{array}{r} \\ \\ 40 \\ \\ 8 \\ \hline 48 \end{array}$

$\therefore\ 342 \div 7 = 48\frac{6}{7}$ $\therefore\ 342 = (48 \times 7) + 6$

4. $n \times b \leq a$ in order to assure that the multiple of b is less than or equal to a.

If $n \times b > a$, the subtraction would not be meaningful.

$r < b$ in order to be sure that n is as large as it can be.

If $r = b$, the quotient would be one more than n;

if $r > b$, the quotient would be at least one more than n with or without a remainder.

CHAPTER 16

1. d. and e. only

2. a. one pound ; b. 18 ; c. 18 pounds

3. a. 9 chalk pieces; b. 9 ; c. one chalk piece

4. c. and e. only

5. a. 1 , 1 , 1
   b. 45 ; 4
   c. 4 is not the sum of the measures of the sides in feet.
   d. The measure of a perimeter of a polygon is obtained by the measure of a segment which is the union of non-overlapping segments congruent to the sides of the polygon. Each side of the triangle is longer than one foot, and therefore the errors account for the extra foot in the perimeter.

6. The measure of $\overline{CD}$ is 1 . The measure of $\overline{EF}$ is 1 . No. No. Congruent segments must have the same measure, regardless of the unit. However, segments may have the same measure without being congruent. It is necessary, however, that with reference to some unit, non-congruent segments must have different measures. In the case of $\overline{CD}$ and $\overline{EF}$, the measure of $\overline{CD}$ is 6 and the measure of $\overline{EF}$ is 8 if the unit is G — H

GLOSSARY

Mathematical terms and expressions are frequently used with different meanings and connotations in the different fields or levels of mathematics. The following glossary explains some of the mathematical words and phrases as they are used in this book and in the K-3 texts. These are not intended to be formal definitions. More explanations, as well as figures and examples, may be found in the book.

A

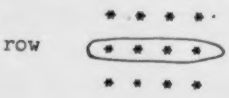
ADDEND. If 8 is the sum of 2 and 6, then 2 and 6 are each an addend of 8.

ADDITION. An operation on two numbers, called addends, to obtain a unique third number called their sum.

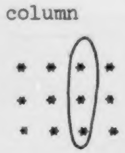
ALGORITHM. A numerical expression of a computation using properties of addition and multiplication and characteristics of a system of numeration to determine the standard name for a sum, difference, product, or quotient.

ANGLE. The union of two rays which have the same endpoint but which do not lie in the same line.

ARRAY. An orderly arrangement of rows and columns which may be used as a physical model to interpret multiplication of whole numbers. For example,



3 × 4



3 × 4

A rectangular array is implied by ARRAY unless otherwise specified.

AS MANY AS; AS MANY MEMBERS AS. If two sets are equivalent, then one set is said to have as many members as the other set.

ASSOCIATIVE PROPERTY OF ADDITION. When three numbers are added in a given order, the sum is independent of the grouping. That is, for any three numbers a, b, and c,

$$(a + b) + c = a + (b + c).$$

ASSOCIATIVE PROPERTY OF MULTIPLICATION. When three numbers are multiplied in a given order, the product is independent of the grouping. That is, for any three numbers a, b, and c,

$$(a \times b) \times c = a \times (b \times c).$$

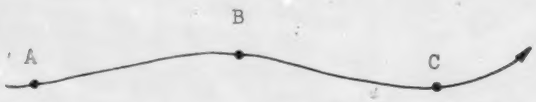
B

BASE (of a geometric figure). A particular side or face of a geometric figure. For example, the base of a parallelogram is one of the sides; the base of a square pyramid is the face that is the square region.

BASE (of a numeration system). A basic number in terms of which we affect groupings within the system. Ten is the base of a decimal system and two is the base of a binary system.

BASIC FACTS (addition, multiplication, subtraction, division). Basic addition and multiplication facts are sentences which express two names for the sums and products of all ordered pairs of whole numbers less than 10 One name expresses the sum or product, using the ordered pair. The other name expresses the sum or product, using the standard name. For example, $2 + 4 = 6$ is a basic addition fact; $3 \times 4 = 12$ is a basic multiplication fact. Basic subtraction and division facts express the differences and quotients for any ordered pairs of whole numbers a and b, such that $a - b = c$ if $c + b = a$ and $a \div b = c$, such that $c \times b = a$, where b and c are both whole numbers less than 10.

BETWEEN. If a curve passes through three points A, B, and C,



then B is between A and C When a curve is not specified, it is understood that the curve is a line or a segment through the points.

If for three numbers a b, and c, $a < b$ and $b < c$, then b is between a and c.

BINARY OPERATION. See OPERATION.

C

CARDINAL NUMBER. See NUMBER PROPERTY OF A SET.

CARTESIAN PRODUCT. If, for two given sets, $A = \{a, b, c\}$ and $B = \{1, 2\}$, then the Cartesian product (product set) of A and B is expressed as

$$A \times B = \{(a, 1), (a, 2)\ (b, 1), (b, 2), (c, 1), (c, 2)\}.$$

CIRCLE. The set of all points in a plane which are the same distance from a given point. Alternatively, a circle is a simple closed curve having a point O in its interior such that, if A and B are any two points of the circle, $\overline{OA}$ is congruent to $\overline{OB}$.

CLOSED CURVE. A curve whose starting and endpoints are the same.

CLOSURE PROPERTY of WHOLE NUMBERS UNDER ADDITION. When two whole numbers are added the sum is always a whole number.

CLOSURE PROPERTY OF WHOLE NUMBERS UNDER MULTIPLICATION. When two whole numbers are multiplied the product is always a whole number.

COLUMN. See ARRAY.

COMMUTATIVE PROPERTY OF ADDITION. When two numbers are added, their sum is independent of the order of the addends. For any two numbers a and b, $a + b = b + a$.

COMMUTATIVE PROPERTY OF MULTIPLICATION. When two numbers are multiplied, their product is independent of the order of the factors. For any two numbers a and b, $a \times b = b \times a$.

COMPLEMENT OF A SET. See REMAINING SET.

COMPLETE FACTORIZATION. Factorization of a number into its prime factors. For example $24 = 2 \times 2 \times 2 \times 3$.

COMPOSITE NUMBER. Any counting number other than 1 that is not a prime number.

CONGRUENCE. The relationship between two geometric figures which have exactly the same size and shape.

COORDINATE. The number associated with a point on the number line.

COUNTING. The pairing of objects in a set with the numerals in the equivalent standard set.

COUNTING NUMBERS. Members of $\{1, 2, 3, 4, \ldots\}$; that is, the whole numbers with the exception of 0.

CURVE. A curve is a set of points followed in going from one point to another.

D

DIFFERENCE. The number which is assigned to an ordered pair of numbers under subtraction. 4 is the difference of 6 and 2.

DIGIT. Any one of the numerals in the set {0, 1, 2, 3, 4, 5, 6, 7, 8, 9}.

DISJOINT SETS. Two or more sets which have no members in common.

DISTRIBUTIVE PROPERTY OF MULTIPLICATION OVER ADDITION. A joint property of multiplication and addition. For any three numbers a, b, and c, then

$$a \times (b + c) = (a \times b) + (a \times c).$$

DIVISION. An operation on two numbers, a and b, such that $a \div b = n$ if and only if $n \times b = a$.

DIVISOR. A factor of a number is a divisor of that number. For example, since $4 \times 2 = 8$, 4 and 2 are factors (divisors) of 8.

E

EDGE. The intersection of two polygonal regions which are faces of the surface of a solid. Where two faces meet is an edge of the solid. For cylinders and cones, the boundary of a face is an edge.

ELEMENT. See MEMBER.

EMPTY SET. The set which has no members.

EQUAL. $A = B$ means that A and B are names for the same thing. For example, $5 - 2 = 3$ expresses two names for the difference of 5 and 2; also; $A = B$ if A and B are sets consisting of the same members.

EQUAL SETS. Sets which have exactly the same members.

EQUATION. A sentence which expresses an equality. Open number sentences are called equations if the verb is "equals", or "is equal to".

EQUILATERAL TRIANGLE. A triangle with three congruent sides.

EQUIVALENT. Two or more sets are said to be equivalent if their members can be put into a one-to-one correspondence; that is, each element of A is paired with exactly one element of B and no element of B is left unpaired.

EQUIVALENT FRACTIONS. Fractions which name the same fractional number.

EVEN NUMBER. An integer which can be expressed as $2 \times n$ where n is an integer.

EXPANDED FORM. The numeral 532 written as

$$(5 \times 10 \times 10) + (3 \times 10) + (2 \times 1)$$

or as $$500 + 30 + 2$$

is said to be written in expanded form.

EXTERIOR (OUTSIDE) OF A SIMPLE CLOSED PLANE CURVE. The subset of the plane which excludes both the simple closed curve and the subset of the plane enclosed by the plane geometric figure.

F

FACTOR. If 10 is the product of 2 and 5, then 2 and 5 are both factors of 10.

FEWER THAN; FEWER (MEMBERS) THAN. If, in pairing the elements of A with those of B, there is an element of B which is not paired with any element of A, then A has fewer members than B.

FINITE SET. A set is finite if there is a whole number that will answer the question, "How many elements are there in the set?"

The notation {0, 1, 2, 3, 4, 5, 6} describes the set of the first seven whole numbers, a finite set.

FRACTION. The numeral of the form $\frac{a}{b}$ where b is not equal to 0.

FRACTIONAL NUMBER. See RATIONAL NUMBER.

G

GREATER THAN. Associated with the relation "has more members than" for sets is the relation "is greater than" for numbers. For example, "$9 > 8$" is read "9 is greater than 8". For any two numbers a and b, $a > b$, if $a - b$ is a positive number.

H

HEXAGON. A polygon with six sides.

I

IDENTITY ELEMENT. The number 0 is the identity element for addition because the sum of 0 and any given number is the given number; that is, $0 + a = a$.

The number 1 is the identity element for multiplication because the product of 1 and any given number is the given number; that is, $1 \times a = a$.

IDENTITY PROPERTY. The property which states that there is an identity element under a particular operation.

INFINITE SET. A set is infinite if there is no whole number that will answer the question, "How many elements are there?"

The notation $\{0, 1, 2, 3, 4, 5, 6, \ldots\}$ describes the set of whole numbers, an infinite set.

INTEGERS. Members of the set $\{\ldots -3, -2, -1, 0, 1, 2, 3, \ldots\}$

INTERIOR (INSIDE) OF A SIMPLE CLOSED PLANE CURVE. The subset of the plane enclosed by the simple closed curve.

INTERSECTION. The operation that associates with two sets a third set consisting of elements common to the two given sets. If $A = \{1, 2, 3, 4\}$ and $B = \{2, 4, 6, 8\}$ then $A \cap B = \{2, 4\}$.

INVERSE (DOING AND UNDOING) OPERATIONS. Two operations such that one "undoes" what the other one "does". For example, putting on a jacket and taking it off are inverse operations.

INVERSE UNDER ADDITION. For every integer a there is an inverse ^{-}a such that $a + {}^{-}a = 0$.

INVERSE UNDER MULTIPLICATION. For every rational number $\frac{m}{n}$ different from zero, there is an inverse $\frac{p}{q}$ such that $\frac{m}{n} \times \frac{p}{q} = \frac{1}{1}$.

ISOSCELES. A triangle with two congruent sides.

J

JOIN; UNION. The union of two disjoint sets to form a third set, whose members are all the elements in each of the two sets. For example,

$$\text{if } A = \{\text{red, blue, green}\}, \text{ and } B = \{\text{white, orange}\},$$
$$\text{then } A \cup B = \{\text{red, blue, green, white, orange}\}.$$

L

LENGTH. The common property of congruent segments. We approximate length by measurement or comparison with specified unit segments. In the length approximated by the measurement 5 miles, 5 is the measure and the unit is the mile.

LESS THAN. Associated with the relation "has fewer members than" for sets, is the relation "is less than" for numbers. For example, "$2 < 5$" is read "2 is less than 5". For any two numbers a and b, $a < b$ if $b - a$ is a positive number.

LINE. A line is conceived of as the unlimited extension of a given segment in both directions.

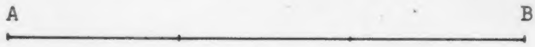
LINE SEGMENT. A special case of the curves between two points. It may be represented by a string stretched tautly between its two endpoints.

LINEAR SCALE. A scale is a number line with the segment from 0 to 1 congruent to the unit being used.

M

MATCH. Two sets match if their members can be put in one-to-one correspondence.

MEASURE. A number assigned to a geometric figure indicating its size (length, area, volume, time, etc.) with respect to a specific unit. For example, the measure in inches of $\overline{AB}$ is 3.



MEMBER (Of a set). An object in a set.

MISSING ADDEND. If 8 is the sum of 2 and n, then n is the missing addend.

MISSING FACTOR. If 10 is the product of 2 and n, then n is the missing factor.

MIXED FORM. See MIXED NUMERAL.

MIXED NUMERAL. A numeral such as $1\frac{1}{2}$ naming a rational number greater than one.

MORE (MEMBERS) THAN. If, in pairing the elements of A with those of B, there is at least one member of B which is not paired with any element of A, then B has more members than A.

MULTIPLICATION. An operation on two numbers to obtain a third number called their product.

N

NATURAL NUMBERS. See COUNTING NUMBERS.

NEGATIVE NUMBER. Any number that is less than 0.

NUMBER LINE. A line marked off at intervals congruent to a chosen unit segment such that: there is a starting point associated with the number 0; the endpoint of successive intervals are labeled according to the counting numbers in their natural order.

NUMBER (PROPERTY) OF A SET. The number of elements in the set. The number property of A is written N(A), where A is a set.

NUMERAL. A name for a number.

NUMERATION SYSTEM. A system for naming numbers. The Roman numeral system and the decimal system are systems of numeration.

O

ODD NUMBERS. An integer which cannot be expressed as $2 \times n$, where n is an integer.

ONE-TO-ONE CORRESPONDENCE. A pairing between two sets A and B, which associates with each element of A a single element of B, and with each element of B a single element of A.

OPERATION. The association of a third number with an ordered pair of numbers is a binary operation. For example, in the operation of addition, the number 7 is associated with the pair of numbers 5 and 2.

In general, an operation is the association of a unique element to each element of a given set, or to each combination of elements, one from each of the given sets.

ORDER. A property of a set of numbers which permits one to say whether a is less than b, greater than b, or equal to b, where a and b are members of the set.

P

PAIRING. A correspondence between an element of one set and an element of another set.

PARTITION. See PARTITIONING.

PARTITIONING. Partitioning a finite set means separating the set into disjoint subsets so that the union of the subsets is the given set.

In partitioning an infinite set such as a line segment, the subsets need not be disjoint. However, any two subsets have at most the points of separation in common.

The separation is the partition.

PATH. See CURVE.

PENTAGON. A polygon with five sides.

PERIMETER(of a POLYGON). The length of the line segment which is the union of a set of non-overlapping line segments congruent to the sides of the polygon.

PLACE VALUE. A value given to a certain position in a numeral. Thus, the place value of the digit 2 in 235 is 100.

PLANE. A particular set of points which can be thought of as the extension of a flat surface, such as the surface of a table.

PLANE REGION. The union of a simple closed plane curve and its interior.

POLYGON. A simple closed curve which is the union of three or more line segments.

PRIME FACTORIZATION. See COMPLETE FACTORIZATION.

PRIME NUMBER. Any whole number that has exactly two different whole number factors, namely itself and 1.

PRODUCT. The third number associated with an ordered pair of numbers by multiplication. For example, 8 is the product of 2 and 4.

PRODUCT SET. See CARTESIAN PRODUCT.

Q

QUADRILATERAL. A polygon with four sides.

QUOTIENT. The third number associated with an ordered pair of numbers by division. For example, 12 is the quotient of 48 and 4.

R

RATIO. A relationship between an ordered pair of numbers a and b where $b \neq 0$. The ratio may be expressed by $a : b$ or by $\frac{a}{b}$.

RATIONAL NUMBER. A number which may be expressed as $\frac{a}{b}$ or $-\frac{a}{b}$, where a and b are whole numbers with $b \neq 0$.

RAY. Ray AB is the union of segment AB and all points C such that B is between A and C.

RECTANGLE. A quadrilateral with four right angles.

REGION. See PLANE REGION AND SOLID REGION.

REMAINDER; REMAINDER SET. See REMAINING SET.

REMAINING SET; REMAINDER (SET). If B is a subset of A, all members of A which are not members of B are members of the remaining or remainder set. The complement of B relative to A is the remaining set.

RENAMING. Using another name for the same number. For example, 34 can be renamed as 30 + 4, 20 + 14, 2 × 17, and so on.

RIGHT ANGLE. One of two congruent angles determined by a line and a ray having a point in the line as endpoint.

RIGHT TRIANGLE. A triangle with one right angle.

ROUND. A shape which has no corners or sides.

ROW. See ARRAY.

RULER. A straightedge on which a scale using a standard unit has been marked.

S

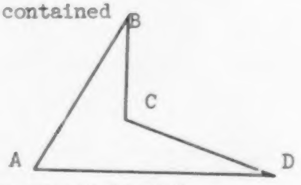
SCALE. See LINEAR SCALE.

SEGMENT. See LINE SEGMENT.

SENTENCE. A statement, such as "9 + 5 = 14" is a number sentence; it connects sets of numerical and operational symbols showing a relation between the sets of symbols. Examples of symbols relating the sets are: =, <, and >. These symbols act as verbs in the sentences.

SIDE (OF AN ANGLE). Each of the two rays forming the angle is called a side of the angle.

SIDE (OF A POLYGON). A segment of a polygon that is contained in no segment of the polygon other than itself. For example, $\overline{AB}$, $\overline{BC}$, $\overline{CD}$, and $\overline{DA}$, are sides of the quadrilateral illustrated at the right.



SIMPLE CLOSED CURVE. A closed curve which does not intersect itself.

SIMPLEST FORM. A fraction is said to be in simplest form when the greatest common factor of its numerator and denominator is 1.

SOLID. A geometric figure that is not a subset of any one plane.

SQUARE. A rectangle whose sides are congruent.

STANDARD SET. One of the sets of ordered numerals such as {1, 2, 3, 4}, {1, 2, 3, 4, 5}.

STANDARD UNIT. A standard unit is a unit of measure "officially" agreed upon or accepted as a standard. Examples are: inch, meter, gram.

SUBSET. Given two sets A and B, B is a subset of A if every member of B is also a member of A.

SUBTRACTION. An operation on two numbers a and b to obtain a third number n, called the difference such that $a - b = n$ if $n + b = a$.

SUM. The third number associated with an ordered pair of numbers by addition. For example, 6 is the sum of 2 and 4.

T

TIMES. The word associated with × to indicate the operation, multiplication.

TRIANGLE. A polygon with three sides.

U

UNION. The operation that associates with two sets a third set consisting of all the members in each of two sets. For example,

if A = {red, blue, green, white, yellow} and
B = {blue, white, orange},
then $A \cup B$ = {red, blue, green, white, yellow, orange}.

UNIT. A prototype from which the measure is obtained by comparison. For example, the unit in measuring length is a segment; the unit for area is a square region.

UNIT REGION. See UNIT.

V

VERTEX OF AN ANGLE. The common endpoint of its two rays.

VERTEX OF A POLYGON. If two sides have a point in common then this commpon point is a vertex. The plural of vertex is vertices.

VERTEX OF A PRISM OR PYRAMID. If three or more edges have a point in common, then the common point is a vertex.

W

WHOLE NUMBER. The property common to a set of equivalent sets. Members of {0, 1, 2, 3, ...}.

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